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Zoology. — *The occurrence of eggs in one of the testes of a Rhizocephalan.*
By H. BOSCHMA.

(Communicated at the meeting of November 26, 1949.)

In a specimen of a Rhizocephalan parasite of the crab *Paraxanthus barbiger* (Poëppig), collected at Talcahuano at the coast of Chile, the left testis contains a large number of eggs that in every respect correspond with those found in the ovary. As the specimen belongs to a hitherto undescribed species a description of its characters is given here.

***Loxothylacus armatus* nov. spec.**

Talcahuano, Hassler Expedition, 1 specimen (holotype) detached from *Paraxanthus barbiger* (Poëppig), collection Museum of Comparative Zoölogy, Cambridge, Mass., no. 1245.

Specific characters. Male genital organs in the posterior half of the visceral mass. Testes completely separated, the left much larger than the right. Right testis curved in anterior direction, left testis enlarged into a voluminous pouch extending in anterior direction. Colleteric glands slightly before the centre of the visceral mass, with more than 100 canals in their most strongly branched region. External cuticle of the mantle with hairs which have a length of 15 to 25 μ , and which bear minute lateral hairs. Internal cuticle of the mantle with retinacula consisting of a flat basal part and 20 to 35 spindles. The latter are not barbed, they vary in length from 9 to 18 μ .

The specimen (fig. 1) has a greater (dorso-ventral) diameter of $16\frac{1}{2}$ mm, an antero-posterior diameter of 14 mm, and a smaller diameter ("thickness") of $5\frac{1}{2}$ mm. It has a roundish pentagonal shape. The mantle opening, surrounded by a strongly developed sphincter, is found at the left side of the anterior margin. With the exception of a broad groove in the centre of the posterior half of the right side (caused by pressure of the median ridge of the abdomen of the host) the mantle does not show any pronounced grooves or wrinkles.

Longitudinal sections show that the male genital organs are contained in the posterior half of the visceral mass. Parts of five sections are drawn, one (fig. 3 a) showing the ventral parts of the vasa deferentia, the next (fig. 4 a) is at a distance of 900 μ farther dorsally, whilst the distance from 4a to 4b is 30 μ , that from 4b to 4c 75 μ , that from 4c to 4d 225 μ . The ventral part of the two vasa deferentia is narrow (fig. 3 a); the right does not become perceptibly wider (fig. 4 a—c), and gradually passes into the right testis which extends for some distance in an anterior direction (fig. 4 d). The left vas deferens towards the dorsal region

gradually increases in size (fig. 4 a), towards its dorsal extremity it passes into a voluminous oval shaped pouch extending in an anterior direction.

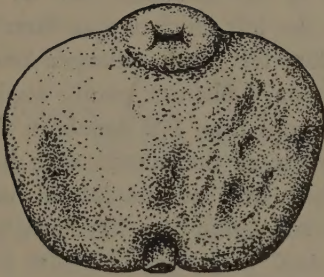


Fig. 1. *Loxothylacus armatus* nov. spec.
Left side. $\times 2\frac{1}{2}$.

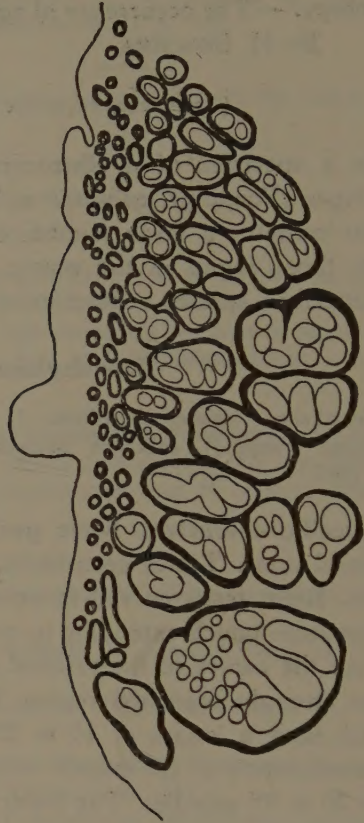


Fig. 2. *Loxothylacus armatus* nov. spec.
Section of the right colleteric gland.
Posterior region in the upper part of
the figure. $\times 60$.

The antero-posterior dimension of this pouch is about twice its ventro-dorsal dimension. Its ventral part is visible in fig. 3 a, its dorsal part in fig. 4 d.

The colleteric glands lie slightly before the middle region of the visceral mass. In their most strongly divided region a longitudinal section shows at least 100 canals (fig. 2). The figure shows, drawn with thick lines, sections of 102 canals with the epithelium that secretes the chitinous matter of the colleteric gland. In many of these canals there are one or more, sometimes a great number of tubes drawn with thin lines, representing the chitinous matter secreted by the epithelium. Apparently in the present specimen the chitinous canal system was already partly withdrawn from the tubes in which it was secreted, so that the narrower canals do no more contain any chitinous matter, and in many of the larger canals there are

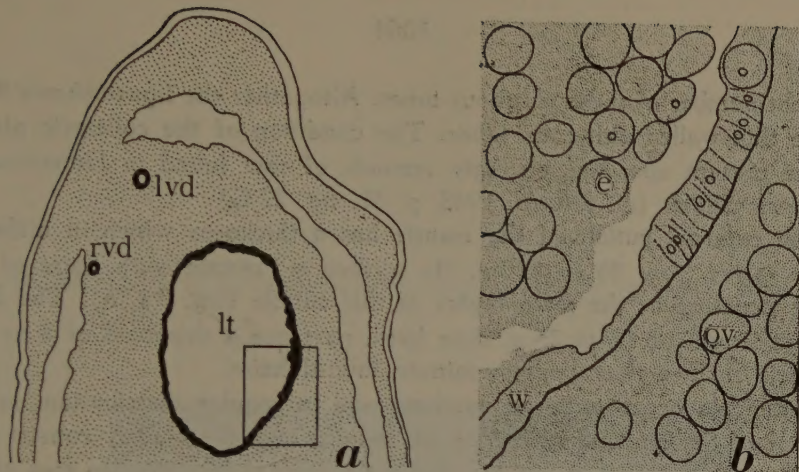


Fig. 3. *Loxothylacus armatus* nov. spec. a, posterior part of a longitudinal section showing the ventral parts of the male organs; b, the region indicated in a by the rectangle; c, eggs in testis; lt, left testis; lvd, left vas deferens; ov, eggs in ovary; rvd, right vas deferens; w, wall of testis. a, $\times 12$; b, $\times 54$.

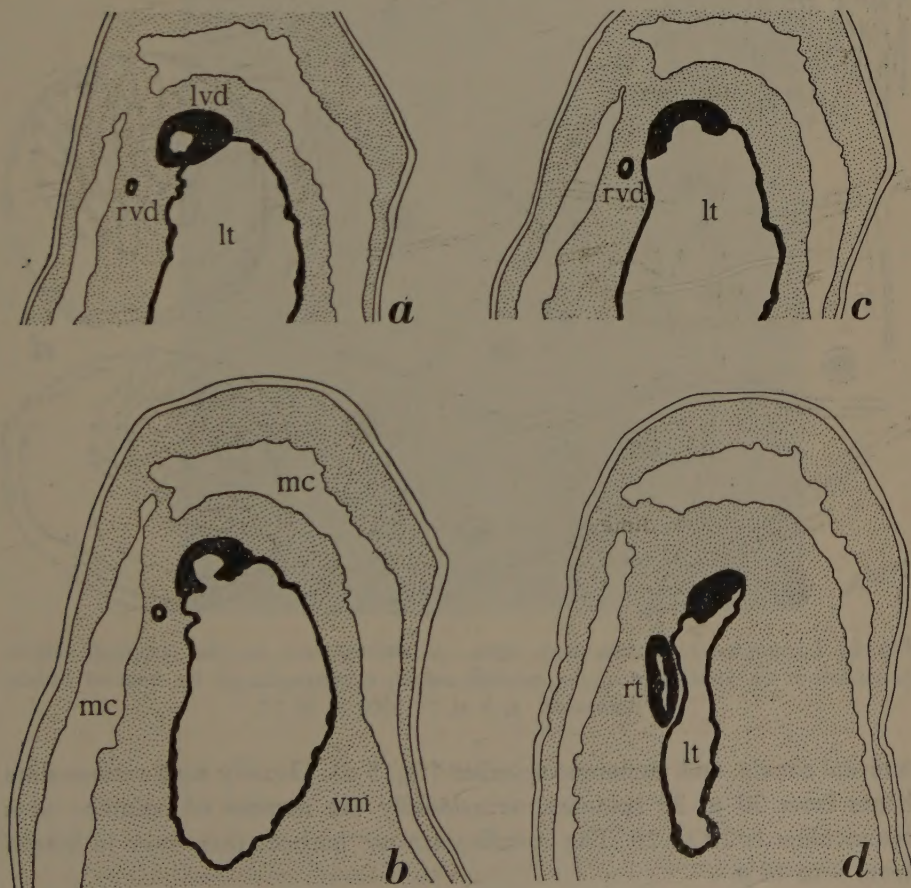


Fig. 4. *Loxothylacus armatus* nov. spec. Posterior parts of longitudinal sections, a from a more dorsal region than fig. 3a, each following section from a still more dorsal region; lt, left testis; lvd, left vas deferens; mc, mantle cavity; rt, right testis; rvd, right vas deferens; vm, visceral mass. $\times 12$.

p. 370 and fig. 65). In *L. sclerothrix* the retinacula have 15 to 30 spindles, which, however, are considerably smaller (4 to 8 μ) than those of *L. armatus*. Moreover the two species have entirely different excrescences of the external cuticle. The trivial name *armatus* was chosen to emphasize the peculiarities of the retinacula.

In many parts of the pouch-like extension of the left testis the wall is composed of rather high cells of various thickness. In the centre of most of these cells a nucleus is found surrounded by a varying amount of yolk, the largest of these have the size of normal eggs in the ovary. A part of the testis showing this formation of eggs is reproduced in the photograph on the plate accompanying the present paper; in a more diagrammatic manner the same part is represented in fig. 3 b. Both figures show that inside the testis there is a large mass of closely packed eggs, most of which clearly show a round contour, and many of which have a distinct nucleus. In their structure and affinity to stains these eggs in the testis do not present any differences from the normal eggs in the surrounding ovary. In the preserved specimen the mass of eggs, obviously on account of shrinkage, has detached itself from the wall of the testis, so that an open space occurs which undoubtedly is abnormal.

It is difficult to find an explanation of the occurrence of eggs in the testis of a Sacculinid. It seems to serve no purpose that, next to the ovary fully packed with eggs, there are developing eggs of normal appearance within the larger of the two testes. No similar phenomena in invertebrates that are normally hermaphroditic in so far that male and female organs are functioning at the same time seem to have been observed. Prof. G. J. VAN OORDT, Utrecht, kindly drew my attention to a phenomenon resulting into a corresponding state of reversal of sexual functions in molluscs, where eggs and sperm are produced in the same follicle of the ovotestis. A striking example is that of *Arianta arbustorum* (L.), as studied by BURESCH, and referred to in the handbook by SIMROTH and HOFFMANN (1908—1928, pp. 644 et seq., figs. 205 A and B). In these protandric snails as a normal phenomenon eggs and sperm are produced in the same follicle. At first the eggs are found in the wall of the follicle only (loc. cit., fig. 205 A); in later stages, when the male cells are degenerating, the eggs are found in the cavities of the follicles (loc. cit., fig. 205 B). But here it is of normal occurrence that in the same organ the production of male cells is followed by the production of female cells, whilst in Sacculinidae the testes, even in old specimens, never are seen to produce eggs.

Unfortunately but one specimen of *Loxothylacus armatus* is available, so that there is no occasion to study the male glands of other specimens. Owing to its large size it may be safely assumed that the specimen is comparatively old, but this does not explain the aberrant phenomenon in its male organs.

I am indebted to Prof. S. T. BOK for the photograph on the plate.

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Aerodynamics. — *Solutions of the equations for the non-uniform propagation of a very strong shock wave. II.* By W. P. ROBBERTSE and J. M. BURGERS. (Mededeling no. 63 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.) *)

(Communicated at the meeting of November 26, 1949.)

6. *Comparison of the solution obtained in sections 4 and 5 with the solutions given in the thesis for the case $g = 0$.* — In the thesis the main attention had been directed to the case of constant density of the gas penetrated by the advancing shock wave. When the density is constant, solutions in which it is assumed that the function Φ has the form (12) cannot be made exactly to satisfy both the equation of motion and the condition to be fulfilled at the shock wave. In order to obtain an approximate picture, two alternatives had been considered: the solution was made either to fit exactly the equation of motion, satisfying the boundary condition at the shock wave in two points only; or else, it was made to fit exactly the boundary condition and satisfied the equation of motion only for a single value of s . It was found that the two solutions obtained in this way did not widely differ (compare e.g. thesis p. 84, fig. 12), so that the result of the approximation did not seem unsatisfactory. A certain freedom was left, since one could shift the values of s for which the remaining condition was satisfied. In the case of the solution exactly satisfying the equation of motion one of these points was always chosen at the origin (determined by $u = v = 1$); the second one could be taken either likewise in the origin, or be moved to some other position. When the second point was taken also at the origin, it was found that the parameter λ should have the value $\sqrt[3]{7/8}$.

In the present paper we have found a solution exactly satisfying both the equation of motion and the boundary condition at the shock wave, but requiring a variable density. We can now attempt to dispose of the arbitrary constants occurring in the solution in such a way that $d\rho_0/ds$ becomes zero for two values of s . If we take both points at the origin, it can be expected that the solution will come most near to those considered in the thesis. Expressed mathematically the condition to be satisfied takes the form:

$$d\rho_0/ds = 0 \quad ; \quad d^2\rho_0/ds^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

As starting point of the motion we take the value of t corresponding to $u = v = 1$; the equations (25) then apply to $v = 1$.

*) Continued from these Proceedings, p. 965.

We shall not immediately introduce the value (29) for λ^2 , but make use of (28). This gives:

$$\varphi_1 = \frac{7u^3(1+\lambda^2)}{3v_1'\lambda^2} - at. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

In this expression u is a function of the time, determined by (15). Differentiating with respect to t and having regard to (15a) and (27) we find:

$$V = \left(\frac{\partial \varphi}{\partial t} \right)_1 = \frac{\sqrt{E}}{3v_1'\lambda^2} \left\{ 7(1+\lambda^2) \frac{\sqrt{u^2+\lambda^2}}{u} - (7+4\lambda^2) \sqrt{1+\lambda^2} \right\}. \quad (32)$$

If we take $u = 1$, we obtain:

$$V_0 = \frac{\sqrt{E} \sqrt{1+\lambda^2}}{v_1'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32a)$$

which expression represents the velocity of the body at the moment it struck upon the gas.

Differentiating (32) with respect to t we find:

$$\frac{dV}{dt} = - \frac{7E(1+\lambda^2)}{9v_1'u^5} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

On the other hand, from eq. (7) with the expression for p given in section 5 and equation (26) for ϱ_0 , all taken for the case $v = 1$, we derive the following formula for the pressure in the points of the path $v = 1$:

$$p = \frac{4E\varrho_{00}v_1'}{3\sqrt{1+\lambda^2}u^5} \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Inserting (33) and (34) into (30) we arrive at the relation:

$$\frac{\bar{\varrho}}{\varrho_{00}} = \frac{12}{7} \frac{v_1'^2}{(1+\lambda^2)^{3/2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

If $\bar{\varrho}$ and ϱ_{00} are supposed to be given, this equation determines the value of v_1' , for any value of λ^2 that may have been chosen. If also V_0 has been given, eq. (32a) fixes the value of $\sqrt{E}$. The values of α and β are then obtained from (27) and (28).

The parameter a , which played an important part in the thesis, is not necessary in these considerations. When it is calculated, e.g. from eq. (3.4), p. 13 of the thesis, its value is found to be $\frac{3.6}{7}(\varrho_0)_1/\bar{\varrho}$, where $(\varrho_0)_1$ is the value of ϱ_0 for $v = 1$. This corresponds to the value given on p. 70 of the thesis. It is also found that $v_1' = a/3$. These relations are independent of the particular value of λ^2 .

9. The course of V as a function of t follows from (32), when use is made of the relation (15) between u and t . The constant E can be eliminated by dividing (32) by (32a), which gives:

$$\frac{V}{V_0} = \frac{1}{3\lambda^2} \left\{ 7\sqrt{1+\lambda^2} \frac{\sqrt{u^2+\lambda^2}}{u} - (7+4\lambda^2) \right\} \cdot \cdot \cdot \quad (36)$$

Since u goes to infinity when t increases without limit, we find:

$$\lim_{t \rightarrow \infty} \frac{V}{V_0} = \frac{7\sqrt{1+\lambda^2} - (7+4\lambda^2)}{3\lambda^2} \cdot \cdot \cdot \cdot \cdot \quad (36a)$$

This quantity is negative, meaning that the direction of motion of the body has been reversed. With $\lambda^2 = 7/8$ the limiting value of V/V_0 becomes -0.35 .

In the thesis the course of V , considered as a function of u , had not been given for $\lambda^2 = 7/8$, but only for $\lambda = 0$, $\lambda = 1$, $\lambda = \infty$. For the case $\lambda = 1$ the following Table gives a comparison between the three solutions:

u	$\bar{V}_1/\bar{V}_0$	$\bar{V}_1^*/\bar{V}_0$	V/V_0
1	1	1	1
1,2	0,629	0,629	0,628
1,4	0,385	0,386	0,385
1,6	0,213	0,215	0,220
1,8	0,082	0,090	0,091
1,9	0,032	0,039	0,032
2,0	-0,014	-0,006	0,022
3	-0,286	-0,270	-0,189
4	-0,397	-0,387	-0,265
6	-0,493	-0,493	-0,321
8	-0,534	-0,540	-0,341
10	-0,551	-0,567	-0,351
∞	-0,590	-0,645	-0,367

The first column is taken from the thesis Table XXI (p. 81) and refers to a solution constructed for constant ϱ_0 , exactly satisfying the equation of motion; the second column, taken from the thesis, Table XXII (p. 82), refers to a solution likewise constructed for constant ϱ_0 , but exactly satisfying the condition at the shock wave; the third column has been calculated from (36). It will be seen that until the instant of reversal of the direction of V , the three solutions differ very little, so that it may be expected that a good approximation is obtained for this phase of the process.

It has already been mentioned in section 5 that the value of ϱ_0 as given by (26) ultimately decreases proportionally to $x^{-7/4}$.

10. We now will give some attention to the solution obtained with $\lambda = 0$. This solution had already been found at p. 22 of the thesis; in section 5 of the present paper it has been mentioned that it also represents the asymptotic form of any solution with $\lambda > 0$. It can easily be derived in a direct way if we take:

$$\varphi = t \cdot w(s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where the function $w(s)$ is unknown. The assumption of a linear dependence on the time, made for φ , has the effect that the velocity of every individual element of volume becomes constant. Hence in the x, t -diagram the paths of the various elements of volume are straight lines, starting from the trace $x = X(t)$ of the shock wave. When the phenomenon is produced by the motion of a body or perhaps of a dense cloud, penetrating into the gas at rest, the velocity of this object must likewise be a constant. This will require that the mass of the body (or its density per unit of frontal area) shall be high.

Since

$$\partial \varphi / \partial s = \Phi = t \cdot w' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

the trace of the shock wave is determined by $w' = 1/t$. Along this trace:

$$S' = ds/dt = -w'^2/w'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

The condition (17) consequently gives

$$w = -3w'^2/w'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

the integral of which is

$$w = C s^{1/4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

It follows that $w' = \frac{1}{4} C s^{-3/4}$ and since along the trace of the shock wave also $w' = 1/t$, we obtain:

$$s = (Ct/4)^{4/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and:

$$X = 4(Ct/4)^{4/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The front velocity of the shock wave consequently is:

$$\xi = \frac{4C}{3} \left(\frac{Ct}{4} \right)^{1/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43a)$$

Since the velocity of every element of volume is constant, the pressure gradient in the gas must be zero. This requires that at every instant the pressure $p = p_1 \Phi^{-5/3}$ shall be independent of s . Now p_1 is proportional to $\varrho_0 S'^2$; hence it follows that ϱ_0 must be proportional to

$$w'' \cdot (w')^{-4} \cdot (w')^{5/3} = w'' \cdot (w')^{-7/3}.$$

As this is proportional to $s^{-7/4}$ we finally obtain $\varrho_0 \propto X^{-7/4}$.

11. To have a numerical example, we take $4(C/4)^{1/3} = 10^6$ (c-g-s-units). This gives:

$$X = 10^6 t^{1/3} \text{ cm.}$$

The front velocity of the shock wave becomes

$$\xi = \frac{4}{3} \cdot 10^6 t^{2/3} \text{ cm/sec,}$$

and the velocity of an element of volume of the compressed gas:

$$u_1 = \frac{3}{4} \xi = 10^6 t^{1/3} \text{ cm/sec,}$$

where t is the time at which this element was reached by the shock wave. As was mentioned before, after the passage of the wave each element keeps the velocity it has required without change.

If we assume that the process is started at $t = 1$ sec, the body or cloud which initiated the shock wave must have a constant velocity of 10 km/sec. The velocity of the shock wave is initially 13.3 km/sec; at the time $t = 1000$ sec this velocity will have increased to 133 km/sec. The distance travelled has then increased from $X = 10$ km to $X = 100.000$ km.

The ideal temperature at the front of the shock wave is given by:

$$RT_{\text{front}} = p_1/\rho_1 = \frac{1}{3} u_1^2 = \frac{1}{3} 10^{12} t^{2/3}.$$

With a mean molecular weight equal to 1, R has the value $8.3 \cdot 10^7$ and we find:

$$T_{\text{front}} = 4000 t^{2/3}.$$

In the period considered this temperature would rise from 4000° to 400.000° . It will be evident that in any actual case the processes involved in the mechanism of collision (ionisation and radiation) and conduction of heat, should be considered, before a calculation of the real temperature would be possible.

In consequence of the expansion the ideal temperatures in the gas behind the wave front are lower than T_{front} . They can be found from the equation:

$$T = T_{\text{front}} \cdot \Phi^{-2/3}.$$

If for clearness we write t_1 for the instant at which an element of volume is reached by the shock wave, and t for any later instant, we have:

$$T_{\text{front}} = 4000 t_1^{2/3} ; \quad \Phi = t/t_1 ; \quad \text{hence} \quad T = 4000 t_1^{2/3} / t^{2/3}.$$

In order to find the temperature distribution in space, we observe that at the instant t_1 the element found itself at $X_1 = 10^6 t_1^{1/3}$, and that in the interval from t_1 to t it has travelled with the constant velocity $u_1 = 10^6 t_1^{1/3}$. Hence at the instant t this element finds itself at the position

$$x = 10^6 t \cdot t_1^{1/3} = X_1 \cdot t/t_1 = X \cdot (t_1/t)^{1/3}.$$

At any instant t_1 the temperature distribution behind the shock wave front consequently can be described by the formula:

$$T = 4000 \cdot t_1^{1/3} \cdot (x/X)^2.$$

12. When gravity is acting with a constant value of g , we take

$$\varphi = t \cdot w(s) - \frac{1}{2} g t^2.$$

Formulae (38) and (39) remain unchanged, while (40) is replaced by:

$$w - \frac{g}{w'} = - \frac{3w'^2}{w''}.$$

Its integral can be expressed in parametric form by the equations:

$$w = \gamma y - \frac{1}{2} g y^3 \quad ; \quad s = \frac{1}{4} \gamma y^4 - \frac{1}{4} g y^6,$$

where γ is a constant, corresponding to $4(C/4)^{1/3}$ in the preceding case. We then have $w' = y^{-3}$, and we obtain $y = t^{1/3}$ at the trace of the shock wave, along which again $w' = 1/t$. In the notation used in sections 3 and 5, applied to the case $\lambda = 0$, we have $u^3 = t/E$ from (15); hence it will be seen from (12) that $v^3 = \sqrt{E}/w' = y^3 \sqrt{E}$.

Along the trace of the shock wave we find:

$$s = \frac{1}{4} \gamma t^{4/3} - \frac{1}{4} g t^2 \quad ; \quad X = \gamma t^{4/3} - g t^2,$$

and

$$\xi = \frac{4}{3} \gamma t^{1/3} - 2 g t.$$

The following expression is now obtained for the density:

$$\rho_0 = \frac{\rho_{00}}{y^7 (1 - 3 g y^2 / 2 \gamma)^2}.$$

The solution can be used only so long as $y^2 < 2\gamma/3g$, which means that in describing the trace of the shock wave we must restrict to:

$$t < (2\gamma/3g)^{3/2}.$$

It will be seen that ξ goes to zero when this limit is approached.

In the numerical example of the preceding section we took $\gamma = 10^6$; hence in order that the expressions obtained in that section may give a tolerable approximation for $t = 1000$ sec, it would be necessary that g should be sufficiently small in comparison with $2\gamma/300 \cong 6700$.

Potchefstroom (Transvaal)-Delft, June-November 1949.

Résumé.

Une solution en termes finis a été développée pour un problème ayant trait à la propagation d'une onde de choc à vitesse variable. Cette solution

ne s'applique que dans le cas où la densité du milieu décroît suivant une loi déterminée. On compare cette solution avec celles étudiées par l'un des deux auteurs (W. P. ROBBERTSE) dans sa thèse de doctorat (Delft 1948). La vitesse de l'onde de choc augmente au cours de la propagation. L'exposé d'un exemple simple fait ressortir que de tels cas peuvent avoir un intérêt dans l'astrophysique.

Quelques détails sont donnés concernant l'influence de la pesanteur sur la propagation.

Resumo.

Oni donas solvon en finitaj termoj de problemo rilata al la propago de sku-ondo kun ne-konstanta rapido. Ĉi tiu solvo estas aplikebla nur en la kazo, ke la denseco de la medio malkreskas laŭ difinita leĝo. La solvo estas komparata kun solvoj studitaj de unu el la aŭtoroj (W. P. ROBBERTSE) en sia doktora disertaĵo (Delft 1948). La rapido de la ondo kreskas dum la propagigo. Pritrakto de simpla ekzemplo montras ke tiaj kazoj povas havi signifon en la astrofiziko.

Oni donas ankaŭ kelkajn detalojn koncerne la influon de la gravito sur la propagon.

Biology. — *Symmetrie der Levende Wezens.* By J. F. VAN BEMMELEN.

(Communicated at the meeting of November 26, 1949.)

In biologische handboeken, ook van den jongsten tijd, worden de verschillende vormen van symmetrischen lichaamsbouw der Dieren en Planten in een reeks gerangschikt, die, uitgaande van de vormloze gedaante der Amoeben, opklimt tot de bilaterale symmetrie der „hogere” dieren. Het aantal tussentrapen tussen deze beide eindpunten en hun namen worden niet altijd hetzelfde genomen, maar in hoofdzaak komt het neer op de tegenstelling tussen tweezijdige en straalsgewijze symmetrie; anders gezegd *Zygomorphie* en *Actinomorphie*. Ofschoon aan het verband tussen de verschillende configuraties niet veel aandacht wordt geschonken, zo is de bedoeling toch wel duidelijk dat de tweezijdige symmetrie als een eindtoestand te beschouwen ware, die zich uit de straalsgewijze ontwikkeld had ¹⁾. Dit blijkt o.a. ook uit de benamingen regulier en irregulier, die in de plantkunde worden gegeven aan resp. straalsgewijs en tweezijdig symmetrische bloemen, en in de dierkunde aan de twee afdelingen der Echiniden: de Cidariden en de Spatangen: de (schijnbaar) radiair symmetrische en de (duidelijk) tweezijdig symmetrische zeeëgels.

Bij de beoordeling van vormen in de wereld der levende wezens moet men steeds in het oog houden, dat geen dier of plant ook maar een enkel ogenblik aan zichzelf gelijk blijft: niet alleen door den groei, 't zij in toef of in afnemende richting, maar ook door het vermogen om zich te bewegen, te contraheren en zijn gedaante te vervormen, en eveneens door alle andere stofwisselingsverschijnselen. Al die levensuitingen gaan met vormveranderingen gepaard: het meest in het oog lopend zijn deze bij de gedaantewisseling, die tot blijvende uitwendige gestalten aanleiding geeft. Maar zij kunnen evenzeer van voorbijgaanden aard zijn, en in beide gevallen kunnen zij teweegbrengen dat de eigenlijke oorspronkelijke gestalte verborgen wordt. Een sprekend voorbeeld daarvan geven ons de eieren der vogels en reptielen, die uiterlijk sferisch, ellipsoïdisch of ovoïd zijn, maar inwendig een bilateralen bouw bezitten. Hetzelfde geldt voor de cocons der vliegen, de stuifmeelkorrels, in 't algemeen voor rusttoestanden. Houdt men dit in het oog, dan is er grond voor de onderstelling, dat de uitwendige gedaante der levende wezens bedriegelijk kan zijn, en dus niet zonder meer uitsluitsel kan geven over de eigenlijke geaardheid en de verwantschap der organismen. Dit geldt evengoed voor alle andere ken-

¹⁾ Zo schrijft b.v. STEMPELL op blz. 43 van zijn Zoologie: „Man kann folgende Grundformen des Bauplans unterscheiden, die in der angegebenen Weise auch Stammesgeschichtlich aus einander hervorgehen dürften”, en geeft dan de volgorde: irregulär, sferisch, monaxon, radiär, zweistralig, bilateral. Irregulär staat hier voor amorph.

merken: een pootloze hagedis zoals de Hazelworm is daarom nog geen slang, en een slang is evengoed een viervoetig dier als andere van poten voorziene Reptielen. Langs deze en dergelijke overwegingen komt men tot de vraagstelling of niet de reeks der symmetrie-vormen juist in omgekeerde richting moet gelezen worden, zodat de bilaterale bouw als de oorspronkelijkste wordt aangezien, waaruit alle andere door vervorming zich hebben ontwikkeld. Beantwoordt men deze vraag toestemmend, dan komt men vanzelf op de gedachte, dat voor deze veranderingen in gestalte, die zich in meerdere of mindere mate in geheel het dieren- en plantenrijk ten allen tijde moeten voorgedaan hebben, een algemene oorzaak moet bestaan. Die oorzaak meen ik te mogen zien in de vasthechting aan de onderlaag, waarop het organisme steunt, tengevolge waarvan het zich niet meer willekeurig verplaatsen kan, en ook in zijn bewegingen ter plaatse in meerdere of mindere mate wordt belemmerd. Vasthechting moet daarbij opgevat worden in den algemeensten zin: de vogel- of reptielenkiem is vastgehecht aan de eidooier, dus aan een deel van zichzelf; de parasitische *Sacculina* hecht zich vast in het organisme van de krab of kreeft waarin zij binnendringt, de zwevende plankton-organismen (dus die welke niet voorzien zijn van eigen verplaatsingsorganen) zijn in het omgevende water besloten, en worden passief meegesleept. De wortelende planten hechten zich vast aan den grond waarin hunne zaden kiemen.

Gaat men uit van de voorstelling, dat alle levensverschijnselen gebonden zijn aan protoplasten, die oorspronkelijk bipolair georganiseerd waren en zich kruipend voortbewogen op de grens tussen land en water, dan moet men aannemen, dat reeds de eerste levende organismen van den aanvang af een voor- en een achtereind, een rug- en een buikzijde, een rechts en een links bezaten, dus bilateraal symmetrisch waren.

Door zulk een tweezijdig organisme kan men slechts één vlak van symmetrie leggen: het mediane (of sagittale) vlak, en de lijn die men van voor- naar achtereind in dat vlak kan trekken, de lengte-as, is de enige werkelijke (de primaire) symmetrie-as. Alle andere symmetrie-vlakken en de assen, die men zich daarbij denken kan, zijn secundair, en een gevolg van vervorming van oorspronkelijk bilateraal-symmetrische organismen tot radiaal, symmetrische. Bij de gangbare indeling der symmetrieën wordt dit niet aangenomen, maar wordt daarbij aan de vlakken van symmetrie minder aandacht geschonken dan aan de assen. Zodoende is het mogelijk, dat men van assen spreekt, ook waar van symmetrie geen sprake kan zijn, b.v. als men bij een bilateraal symmetrisch organisme naast de lengte-as nog een dorsoventrale en een links-rechtse as aanneemt, welke drie assen loodrecht op elkaar worden gedacht. Vermoedelijk komt dit voort uit het gebruik in de kristallographie om bij de kristallen een meer-assig stelsel aan te nemen, waardoor de loop van de aangroeiing wordt bepaald. Groei door appositie is echter iets heel anders dan groei door intussusceptie, vooral daar de laatste altijd gepaard gaat met differentiatie der toetredende deeltjes in verband met verschillende levensfuncties.

Van een symmetrie-vlak mag men alleen spreken, wanneer daardoor het organisme in helften wordt verdeeld die elkaars spiegelbeeld zijn. Diezelfde eis moet ook gesteld worden voor alle onderdelen van het organisme, daar zij slechts beoordeeld mogen worden in hun samenhang met het geheel. Zo kan men wel spreken van symmetrie tussen linker- en rechterarm (of been), maar niet tussen linker (of rechter) arm en been, al heeft het soms den schijn alsof de extremiteiten van een zelfde zijde een symmetrisch stel vormen. Tot mijn verwondering zag ik dat KEITH in zijn Textbook of Human Embryology betoogt, dat dit wel veroorloofd is, zodat hij instemt met de beweringen van DRENNAN, en diens schematische figuren overneemt. Op een dwaalspoor gebracht door de tegenstelling tussen de elleboog, wier strekzijde naar achteren gekeerd is, en de knie waar dit juist omgekeerd is, zien KEITH en zijn school over 't hoofd, dat die tegenstelling reeds niet meer geldt voor pols- en enkelgewricht, die beide hun buigzij naar achteren keren. Door hun opvatting zien zij zich gedwongen de duim niet te homologiseren met de grote teen, maar integendeel met de kleine — en vice versa —, waarmee weer samenhangt, dat zij het scheenbeen niet gelijkstellen met het spaakbeen, maar met de ellepijp, waarvoor de (schijn) overeenkomst van knieschijf en haakvormig ellebooguitsteeksel van de ellepijp een (bedriegelijke) bevestiging schijnt te leveren.

Bij deze opvatting zou men verder moeten aannemen, dat bij de Zoogdieren niet de voorovergekantelde stand van de hand de meer oorspronkelijke is, maar integendeel de achterovergekantelde, waarbij ellepijp en spaakbeen evenwijdig met elkaar komen te liggen, in stede van over elkaar gekruist. Dat daaruit zou volgen, dat de palmzijde van de hand homoloog ware met de rugzijde van den voet, lijkt mij reeds een afdoend bewijs van de onmogelijkheid om arm en been van eenzelfde lichaamszijde op te vatten als elkaars spiegelbeeld.

Dit wordt ten overvloede nog weerlegd door het vergelijkend-anatomisch onderzoek der beide onderarmbeenderen in de rij der viervoetige dieren, waaruit duidelijk blijkt dat de gekruiste stand van ellepijp en spaakbeen op haar beurt weer ontstaan is uit een nog oorspronkelijker toestand van evenwijdigheid, door opschuiving van het elleboogshoofdje van het spaakbeen naar den buitenkant, over de ellepijp heen, en niet omgekeerd. Daardoor geraakte de ellepijp achter het spaakbeen, en kon bij die dieren, die hun voorpoten niet meer tot grijpbewegingen gebruikten, met den achterkant van het spaakbeen vergroeien onder obliteratie van haar distaal gedeelte. Dat dit niet noodwendig altijd het geval moest zijn, bewijzen de olifanten, wier gekruiste onderarmbeenderen niet met elkaar vergroeien, ofschoon zij hun handen uitsluitend als steunorganen gebruiken. De bouw en de stand der ledematen van die viervoetige zoogdieren, die op hun poten steunen, is alleen begrijpelijk, wanneer men aanneemt dat voor- en achterpoten oorspronkelijk zich in zijwaartse richting uitstrekten en uit die ligging naast de romp omzwiepten naar het midden en tevens in buik-

waartse richting, zodat zij *onder* de romp te staan kwamen, en deze gingen steunen en van den grond opheffen. Daarbij sloegen de achterpoten zich in het heupgewricht naar voren om, zodat hun oorspronkelijke voorrand naar het midden toe gekeerd werd, de voorpoten daarentegen in het schoudergewricht naar achteren, zodat hun voorrand naar buiten gewend werd. Dit verschil in standsverandering was een rechtstreeks gevolg van het onderscheid in beider functie: de achterpoten moesten het lichaam opduwen, de voorpoten het vooruittrekken. Daartoe moesten zich de knieën naar achteren buigen, de ellebogen naar voren, waaruit wederom voortvloeide, dat de spieren aan den voorrand der achterpoten tot strekspieren werden, die aan den voorrand der voorpoten tot buigspieren, wat betekent, dat de homologie van spieren der extremiteiten niet beoordeeld mag worden naar hun functie, maar uitsluitend naar hun oorspronkelijke plaats. De noodzakelijkheid, dat bij deze standverandering der voorpoten de palmvlakte van de hand op den bodem moest blijven rusten, bracht mede, dat de voorrand van den onderarm weer naar het midden gekeerd moest kunnen worden, wat slechts door wringing in het ellebooggewricht bereikbaar was, hetgeen de verschuiving van het elleboogshoofdje van het spaakbeen veroorzaakte.

Zodoende ontwikkelde zich het pro- en supinatievermogen, wat dus niet zonder meer als een oorspronkelijke toestand mag beschouwd worden.

Hoe die dwarsuitstaande ledematen der onbekende voorouders van de Tetrapoden er uit gezien hebben, en hoe zij functioneerden, daarvan kan men zich moeilijk een voorstelling maken. Vermoedelijk waren zij spatelvormig verbreed en afgeplat, en hingen zij tot zekere hoogte met elkaar samen. Het ware echter onjuist te denken dat de gangpoten ontstaan zouden zijn uit de gepaarde vinnen der vissen, veeleer zou men kunnen aannemen, dat juist omgekeerd deze laatste uit de poten van landbewonende viervoeters ontstaan zouden zijn, bij terugkeer in het water, zoals dit met zekerheid het geval is geweest bij de Walvissen, de Zeekoeien en de Robben, en onder de Reptielen bij de Zeeschildpadden, de Ichthyo- en Pleisiosauriers.

Enigszins vermag men misschien zich een voorstelling te maken van den oervorm der ledematen van de viervoetige dieren, wanneer men denkt aan de borstvinnen der Roggen, als men maar nooit uit het oog verliest, dat men bij deze te doen heeft met een toestand van uiterste secundaire wijziging der borstvinnen van de haaien, in stede van met oorspronkelijkheid. Toch zou in de grote vermeerdering der vinstralen in de sterk verbrede borstvinnen der Roggen misschien wel een terugslag mogen gezien worden tot een primitieven toestand, waarin nog alle rompsegmenten aan de vorming van ononderbroken zijplooien deelnamen, zodat deze nog niet in voor- en achterledematen gesplitst waren. Van de wijze waarop die zijplooien bij de voortbeweging in het water functioneerden kan de wijze van zwemmen der Roggen ons enige voorstelling geven, en evenzeer die der Platvissen, al is het bij laatstgenoemden de mediane ongepaarde

vinplooï waarlangs de op- en neergaande golven van kop naar staart verlopen.

Even onjuist als de bij onkundigen gangbare mening, dat de gekleurde oppervlakte der Platvissen de rugzijde, de witte de buikzijde zou zijn, is de opvatting dat de voorste en achterste ledematen der viervoetige dieren in spiegelbeeld-verhouding tot elkander zouden staan. Men zou dan ook wel kunnen betogen, dat voor- en achtereind van den romp zich ten opzichte van elkaar symmetrisch verhielden, omdat aan beider uiteinde een opening van het darmkanaal voorkomt. Wanneer dit werkelijk het geval ware, zouden de wervels in het achterste deel des lichaams juist omgekeerd van die in het voorste deel moeten staan. Seriale homologie is onverenigbaar met spiegelbeeld-symmetrie. Een transversale doorsnee door den romp kan nimmer een symmetrievlak zijn.

In vorige mededelingen ¹⁾ werden reeds enige voorbeelden van den vervormenden invloed der vasthechting besproken, en werd er op gewezen dat deze zich in alle afdelingen van Planten- en Dierenrijk had doen gevoelen, zij het ook in zeer verschillende mate en met ver uiteenlopenden uitslag. De fixatie kan ook tijdelijk zijn, maar altijd zal zij haar sporen achterlaten. Zo is de navel een lidteken van de vasthechting van het zoogdierembryo aan de wand der baarmoeder.

Uit de talloze gevallen waarbij zich die invloed laat bespeuren mogen hier nog enkele nader worden besproken. Dat de vrij zwemmende Kwallen hun straalsgewijze organisatie danken aan oerpolypen, die zich met hun aboraal uiteinde vastzogen en daardoor hun tweezijdige symmetrie deden overgaan in een straalsgewijze, mag men besluiten uit hun verwantschap met Zeerozen (Actinien) en Korallen. Als nalatenschap der bilateraliteit bezitten de Zeerozen nog een spleetvormige mondopening met een invloei-gleuf aan het ene en een uitvloeiporus aan het andere eind. Die uitvloeiporus fungeert dus als anus, en het vlak dat men zich in de lengte van de mondspleet kan denken is eigenlijk het mediaanvlak. De kransgewijze rangschikking der tentakels moet uit een hoefijzervormige scharing ontstaan zijn, en dit openbaart zich nog in de opvolging in aanleg der septa van het kalkskelet, zoals door LACAZE DUTHIERS is bewezen, in tegenspraak met MILNE EDWARDS en HAIME. Even sprekende bewijzen voor de opvatting dat straalsgewijze symmetrie als ontstaan uit tweezijdige symmetrie, levert de andere afdeling der z.g. straaldieren, die der Stekelhuidigen. Bij de beschrijving van hun organisatie wordt meestal uitgegaan van de Zeesterren, en wordt aan deze een zuiver straalsgewijze symmetrie toegekend, ofschoon moet toegegeven worden dat daarop inbreuk wordt gemaakt door de ligging van de madreporenplaat in één der (meestal vijf) tussenstralen. Denkt men zich een verticaal vlak door die tussenstraal,

¹⁾ J. F. VAN BEMMELEN: Dieren en Planten, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 52, 125—132 (1949).

J. F. VAN BEMMELEN: Segmentatie, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 52, 819—828 (1949).

dan wordt de zeester in helften verdeeld die elkaars spiegelbeeld zijn, maar dan vervalt tegelijkertijd de voorstelling van radiaire symmetrie en maakt plaats voor die der tweezijdige. Dit is eveneens het geval wanneer men let op de wijze van kruipen op den zeebodem, waarbij noodzakelijkerwijze hetzij een der armen vooruit moet gaan of wel achteraan moet slepen. Beslissing in deze controverse brengt de ontwikkelingsgeschiedenis: de vrij-zwemmende larven zijn (tenminste uiterlijk) zuiver tweezijdig symmetrisch met een kop- en een staartpool, en bevatten een lichaamsholteblaas met een rechter en een linker excretie-orgaan, die in aanleg gepaard zijn (dipleurula-larve). Maar reeds gedurende het larvenstadium oblitereert een dier excretie-organen, terwijl het andere nephridium tot madraporenplaat en steenkanaal wordt, waardoor de bilateraal-symmetrische bouw wordt verstoord. De bovenvermelde tweezijdigheid van de volwassen zeester volgens een interradiair vlak dat door de madreporenplaat gaat, is dus niet overeenkomstig met die van de larve, maar een gevolg van het uitbotten der armen in een vlak loodrecht op de oorspronkelijke lichaamsas der larve.

Bij het ontstaan van den eersten aanleg van de jonge zeester uit een gedeelte van het larvenlichaam doet zich bij sommige soorten het verschijnsel voor, dat aan dien aanleg een uitwas zich vormt, die wegens zijn stand en zijn vorm niet anders kan begrepen worden dan als homologon van de steel der Haarsterren (Crinoiden). Die uitwas wordt weer gesorbeerd zonder gefunctioneerd te hebben. En nu is het toch wel heel opmerkelijk in de ontwikkeling van Antedon, een steelloze Crinoïd, dat de larve zich met een steel vastzet, op dezelfde wijze als bij de gesteelden het geval is, maar van die steel weer loslaat, en zich vrij gaat bewegen op dezelfde wijze als de Zeesterren.

Uit den loop der ontogenie van de Echinodermen, die in alle afdelingen in hoofdzaak dezelfde is, mag m.i. die hunner phylogenie opgemaakt worden, en wel in dien zin, dat zij afstammen van bilateraal-symmetrische vrij zwemmende organismen, die zich met hun aboraal uiteinde vastzogen en aan dat uiteinde een steel vormden, terwijl hun anus zich verplaatste naar voren, en rondom hun mondopening een krans van (vijf) armen ontstond, waardoor hun tweezijdigheid in straalsgewijzen bouw overging, terwijl uit de ene helft van hun lichaamsholte zich het watervaatstelsel ontwikkelde, wat tot de asymmetrische ligging van de ambulacraalplaat aanleiding gaf.

Ook de definitieve vorming van het kalkskelet moet in verband met den overgang van den zygomorphen in den actinomorphen bouw beoordeeld worden. De straalsgewijze rangschikking der afzonderlijke platen tot een meer of minder samenhangend pantser moet ontstaan zijn door aaneensluiting van aanvankelijk afzonderlijke kalkstaafjes, zoals wij die nog zien optreden in de armen der larven, die volstrekt tweezijdig symmetrisch geschaard zijn. Dat het kalkskelet der Holothuriën (zeekomkommers) beperkt is tot afzonderlijk in de huidspierzak liggende lichaampjes moet

uit reductie van een oorspronkelijk samenhangend pantser verklaard worden. Onder de zee-egels zien wij toch ook bij sommige families de pantserplaten van elkaar losraken (*Asthenosoma*). De uiterlijke tweezijdigheid der Zeekomkommers en der irreguliere Zee-egels is secundair uit de straalsgewijze ontstaan. Zij mag als een terugkeer tot den Zygomorphen bouw worden verklaard.

In het opstel Dier en Plant werd reeds betoogd, dat in 't algemeen de invloed der vasthechting op de organisatie der levende wezens een remmende is, waardoor de verdere differentiatie van hogere vermogens wordt afgesloten, in 't bijzonder die der zintuigen, dus wat men mag noemen de geestelijke ontwikkeling. De maat van dezen invloed hangt m.i. af van den ontwikkelingstrap, waarop de organismen stonden, toen zij zich gingen vasthechten. De nog vrij levende tweezijdig-symmetrische voorouders der Echinodermen moeten reeds een gedifferentieerd zenuwstelsel, darmkanaal, vaatstelsel en excretiestelsel gehad hebben, evengoed als de Tunicaten en in 't algemeen de Vertebraten. Voor de meeste dier orgaanstelsels werd de verdere differentiatie door de fixatie tot staan gebracht, slechts op het vaatstelsel oefende zij bij de Echinodermen een sterken drang tot hogere specialisatie.

Vergelijkt men de Echinodermen met de Tunicaten, dan is de eerste indruk, dat de invloed der vasthechting op elk der beide typen geheel verschillend is geweest. Gaat men uit van de solitaire Ascidiën, dan ontdekt men nauwelijks sporen van straalsgewijzen bouw, maar enkel van een verwringing der tweezijdige symmetrie. Toch mag misschien in de kringsgewijze rangschikking der tentakels in de mondopening een aanduiding van actinomorphen aanleg worden gezien. Meer nog is dat het geval bij de kolonievormende Ascidiën, waarbij zoals in *Botryllus* de enkelvoudige individuen in een rozet rondom de gemeenschappelijke uitvloeioopening gerangschikt staan. Maar deze eigenaardigheden zijn klaarblijkelijk van geheel bijkomstigen aard. Vraagt men naar de oorzaak van dit verschil, dan zou men het misschien kunnen toeschrijven aan het onderscheid tussen de verschillende hoogten van organisatie waarop de vrij levende voorouders der beide diertypen stonden, toen zij tot de vasthechting overgingen. Om met de Tunicaten te beginnen, deze waren reeds volledige Chordaten, zoals door hun nauwe verwantschap met het Lancetvisje (*Amphioxus*) wordt bewezen. Zij hadden dus een ruggstreng en een ruggemerg, een gedifferentieerde kop waarin een hersenblaas met gehoor- en gezichtszintuig, een kieuwkorf in een peribranchiale omhulling, een in metameren gesegmenteerde staart.

Al deze organen werden tengevolge van vasthechting ten dele sterk gewijzigd en gedifferentieerd, of gingen integendeel geheel of grotendeels verloren. Daarbij speelde de doorspoeling met adem- en voedselaanvoerend water de hoofdrol, en deze waterstroom behoefde niet anders dan de reeds aanwezige in- en uitvloeioopeningen. Alleen verplaatste zich de laatste langs de rugzij naar voren tot in de nabijheid van den mond.

De larven der Echinodermen daarentegen zijn niet in microsomiëten gesegmenteerd, maar vertonen enkel het algemene organisatietype der levende organismen, n.l. de indeling in pro-meso- en metasoma. Zij hebben geen chorda, terwijl hun zenuw- en zintuigstelsel geheel in de uitwendige huid is gebleven. De vasthechting, waarbij aan de rugzij een gelede steel zich ontwikkelde, kon dus veel dieper ingrijpenden invloed uitoefenen, terwijl de overgang der hoefijzervormig geschaarde tentakels tot een gesloten krans niet op bezwaren stuitte. Vraagt men nu waarom bij deze zoveel eenvoudiger georganiseerde Echinodermen-larven een zo ingewikkelde metamorfose optrad, die tot zo samengestelde en merkwaardige bouwtypen leidde, dan ziet men zich geplaatst voor het algemene vraagstuk der Evolutie van de levende wezens, dat tot heden onoplosbaar is gebleven. Hier zij dus alleen gewezen op het feit, dat één en dezelfde oorzaak, i.c. vasthechting aan den zeebodem, op verschillend geaarde organismen een geheel verschillende uitwerking heeft gehad. Hetzelfde geldt voor alle andere adaptatie-verschijnselen: het effect der inwerking van uitwendige levensvoorwaarden wordt in hoofdzaak bepaald door den aangeboren aanleg der organismen. Hoe deze daaraan gekomen zijn, blijft even raadselachtig als tevoren.

Ten slotte moet er nog op gewezen worden, dat de beide helften waarin een tweezijdig symmetrisch gebouwd organisme kan verdeeld worden, nimmer volkomen identiek zijn. Voor grote verschillen, zoals bijv. tussen de beide scharen van kreeften of de beide bladhalften der *Begonia*'s, is dit bekend genoeg, maar dat b.v. nimmer de rechter helft van het menselijk gelaat het zuivere spiegelbeeld van de linker is, weten de meeste mensen niet.

Bijzonder duidelijk is de afwijking van de volkomen bilateraliteit te zien in de kleurentekening, niet slechts bij in 't oog lopende verschillen tussen linker en rechter lichaamshelft, maar evenzeer bij ogenschijnlijk geheel overeenkomstige huidtekening, zoals bijv. de streping der zebra's of de vlekken van katten of damherten.

Altijd echter maken deze onvolkomenheden den indruk van secundaire afwijkingen van een oorspronkelijk zuiver symmetrisch grondplan. Zij wekken dus de voorstelling, alsof in het verloop der ontwikkeling (wat men zou kunnen noemen de uitvoering van het plan) de beide helften een invloed op elkander uitoefenen, die van weerszijden verschillend is.

Summary.

All living beings, plants as well as animals, are originally Zygomorphous. The actinormorph condition and all other types of symmetrical habitus have started from the bilaterally symmetrical organisation, under the influence of various lifeconditions, especially fixation to the bottom or to other objects, even to parts of their own body, e.g. the yolk sack. Symmetry of accessory parts of an organism may only be considered in their relation

to the complete individual as a whole; therefore no relation of symmetry can exist between homologous organs of the same bodyhalf, e.g. right arm and leg, as they never can be each other's mirror-image. Serial homology is incompatible with bilateral symmetry. Identical influence of the surroundings on organisms of different constitution leads to widely diverging consequences: e.g. the Echinoderms compared with the Tunicates. In both groups the freeliving adult stages must have been derived from fixed actinomorphic ancestors, which in their turn originated from freeswimming zygomorphic coelomatics.

Résumé.

La zygomorphie est la condition primordiale de l'organisation des êtres vivants. Toutes les autres formes de symétrie sont des dérivés de la bilatéralité. La principale cause de ce changement de configuration est la fixation de l'être aux objets de son entourage. Ces objets peuvent être de différente nature, même des parties de son propre corps, p.e. la fixation de l'embryon des Vertébrés à son sac léctal. La conception de symétrie est indissolublement unie à la possibilité de division de l'individu en moitiés qui sont des images de miroir l'un de l'autre. Par conséquent la symétrie des différentes parties d'un même individu ne peut être jugée qu'en relation avec le corps en son entier.

Il en suit que les membres homologues du même côté d'un organisme zygomorphe comme p.e. le bras et la jambe droite où gauche ne peuvent jamais former un pair symétrique.

L'influence des conditions de vie externes (comme la fixation) sur des êtres de nature différente est extrêmement variable, mais même en tout cas à cacher la zygomorphie originale.

Comme exemple de cette variabilité de l'influence de la fixation les Echinodermes sont comparés entre eux et avec les Tuniciers. En même temps ces deux types d'animaux donnent des preuves du pouvoir des êtres fixées à se délibérer de nouveau et rattraper la bilatéralité, tout en conservant des traces méconnaissables de l'actinomorphie précédente. La symétrie bilatérale des organismes n'est jamais parfaite, mais la déviation de l'état idéal est toujours de nature à prouver son origine d'un tel état.

Zusammenfassung.

Zygomorphe Symmetrie ist die ursprüngliche Organisationsform aller lebenden Wesen; alle andere Gestalten sind daraus hervorgegangen. Hauptursache der Umbildung war die Anheftung an andere Gegenstände, hauptsächlich an den Boden, sondern öfters auch an oder in andere Lebewesen, ja selbst an Teile des eigenen Körpers, wie z.b. am Dottersack. Symmetrisches Verhalten zwischen Unterteile eines und desselben Organismus darf nur in ihrer Beziehung zum Ganzen beurteilt werden. So

können Arm und Bein der selben Körperseite niemals een symmetrisches Paar bilden. Seriale Homologie ist unvereinbar mit Symmetrie.

Identische Lebensbedingungen rufen gänzlich verschiedene Umbildungen hervor bei Organismen verschiedener Beanlagung. So hat die Anheftung an den Meeresboden einen weit anderen Einflusz geübt auf die Echinodermen als auf die Tunicaten.

In zahlreichen Fällen haben sich die festgewachsenen Organismen, Pflanzen ebensogut wie Tiere, wieder losgelöst, aber immer sind die Spuren der Fixation an ihrem Bau erhalten geblieben.

Naschrift.

Op 1 September j.l. was ik juist bezig de slotregels van het voorgaande opstel te schrijven, toen mij het geschrift van mijn diepbetreurden vriend I. C. SCHOUTE: On general Morphology, gebracht werd, waarin het tweede hoofdstuk over de symmetrie der levende wezens bleek te handelen. De eerste indruk was, dat nu mijn artikel wel overbodig zou blijken, dan wel dat het geheel omgewerkt diende te worden. Maar bij nauwgezette kennisneming van den inhoud van SCHOUTE's verhandeling, kwam ik tot het besluit, dat het de voorkeur verdiende mijn artikel geheel onveranderd te laten, maar er alleen enige opmerkingen aan toe te voegen, om duidelijk te doen uitkomen, in welke opzichten wij met elkaar overeenstemden, dan wel van elkaar verschilden. Wat het eerste aangaat, was het mij een grote voldoening te vernemen, dat ook SCHOUTE een absoluut onderscheid maakte tussen mathematische en biologische symmetrie, en dus de laatste geheel onafhankelijk van de eerste behandelde. Ook schonk het mij een gevoel van bevrediging, dat SCHOUTE grote betekenis toekende aan de polariteit der organismen, en de lijn die de kop- met de staartpool verbindt als „main axis” den voorrang toekende boven de andere „biologische” assen, die hij onderscheidde. Daarentegen was het mij een teleurstelling, dat hij diezelfde onderscheiding niet maakte tussen de tweezijdige symmetrie en alle andere vormen van symmetrischen lichaamsbouw, zodat hij niet scheen aan te nemen, dat deze laatste alle uit een oorspronkelijke bilateraliteit zouden ontstaan zijn. Toch zijn in sommige zijner uitspraken wel aanduidingen te vinden, die op een genetisch verband tussen zygomorphie en aktinomorphie wijzen, het duidelijkst in de passage op blz. 64, die ik hier verkort en in 't Hollands vertaald weergeef:

„Slechts twee klassen van symmetrie komen in feite op grote schaal voor: de radiale en de dorsiventrle. In beide treffen wij een hoofdas aan die heteropolair is, en voldoet aan de vereisten zowel van sessiele als van vrijlevende organismen; daar hetzij een van de polen vastgehecht is aan de onderlaag, terwijl de andere vrij blijft, dan wel de ene pool vóór-, de andere achteraan gaat. Van deze twee is de radiale symmetrie het meest geschikt voor orthotrope sessiele organismen, terwijl de dorsiventrle uitstekend voldoet aan de eisen der plagiotrope sessiele en der vrij-

bewegelijke. En als werkelijk de symmetrie der organismen deel uitmaakt van het grondplan, en als zodanig blootgesteld is aan onophoudelijke natuurlijke selectie, mogen wij verwachten dat de geschiktste vormen zullen overwegen, en kunnen wij dus begrijpen, waarom de dorsiventrale symmetrie overheerst in het Dierenrijk, en waarom in dat der Planten beide vormen van symmetrie zeer verspreid zijn."

SCHOUTE zag dus ook verband tussen symmetrische lichaamsbouw en vastzittende of vrijbewegelijke levenswijze. Neemt men nu aan, dat de eerste uit de laatste is ontstaan, dan volgt daaruit vanzelf, dat de radiaire symmetrie uit wijziging der bilaterale is voortgekomen en niet omgekeerd.

Microbiology. — *Selective bacteriostasis by ascorbic acid.* By H. G. DERX.
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(Communicated at the meeting of November 26, 1949.)

In the course of experiments conducted with the view of isolating LINDNER's *Termobacterium mobile* (synonyms: *Pseudomonas Lindneri* KLUYVER et HOPPENBROUWERS (16); *Zymomonas mobile* (Lindner) KLUYVER et VAN NIEL (17)), some rather unexpected phenomena were encountered, which have led to the following considerations.

Termobacterium mobile was first found in 1925 by LINDNER (21) in the sweet, sugar-containing liquid, flowing from the amputated or injured flower-spikes of Agave species in Mexico (*Agave atrovirens* seems to be the principal one). The liquid, called "agua miel", is subject to spontaneous fermentation, which changes it into agave-wine: the popular and famous Mexican "pulqué". The microflora of fermenting agua miel, photographed by LINDNER (20), contains several kinds of organisms. Besides *Termobacterium mobile*, the principal fermenter, several yeasts, lactic-acid ferments and acetic-acid bacteria are present in great number.

According to KLUYVER (15), *Termobacterium mobile* was also found some years ago by ROELOFSEN in fermenting "tuac": the bleeding-juice of the amputated spadix of the Asiatic sugar-palm, *Arenga saccharifera*. It will probably also be present in sugar containing juices of other tropical palms. After fermentation, sweet tuac produces sugar-palm wine, known as "tuac keras" (W. Java), "sagueer" (Moluccas), or "toddy" (Ceylon). During the spontaneous fermentation of tuac the same groups and forms of microbes were found by the present author as those represented by LINDNER. However, fermentation in the liquid collected during the first few weeks after the amputation of the flower-spike was caused by yeasts; whereas tuac collected from the same spike a month later showed a profuse development of *Termobacterium mobile* and relatively few yeasts. The accompanying flora of lactic-acid- and acetic-acid bacteria was, of course, also present; *Leuconostoc* species were found already in the very first portion of tuac collected. By conducting the fermentation under anaerobic conditions (in totally filled stoppered bottles), it was easy enough to get rid of the acetic-acid bacteria. It proved quite difficult, however, to eliminate the lactic-acid ferments, which were present in considerable number and showed a tendency to outgrow *Termobacterium* completely. Dextran-forming lactococci were a particular nuisance in this respect.

In order to facilitate the resumption of the fermentation-process, 0.1 % sodium salt of ascorbic acid was sometimes added with good results to the liquid medium in the stoppered bottles. However, the addition of 0.1 %

sodium ascorbate to the agar medium in Petri dishes proved to have a surprising effect. The presence of this substance in the agar did not lead to an enhancement of the growth of *Termobacterium mobile*, as was expected, but to a complete suppression of the development of lactic-acid bacteria¹). The best results were obtained when a sterilized (and subsequently neutralized) 5 % solution of ascorbic acid was added to the nutrient agar (agar 2.5 %, decoct of germinated seeds of *Phaseolus radiatus*, saccharose 2 %) just before filling the Petri dishes. The agar should be inoculated without delay, though the results are the same if the dishes are kept in a refrigerator during two or three days before inoculation. A longer delay, however, was found to be harmful: neither lactococci nor *Termobacterium* showed any development on sodium ascorbate agar in dishes which had been kept for about a week at $\pm 10^{\circ}\text{C}$.

The complete suppression of growth of the lactic-acid bacteria by a substance such as ascorbic acid was quite unexpected at first sight. In fact, it has been common practise for some years past to add reducing compounds, such as cysteine and especially sodium thioglycollate (BREWER (4)), to nutrient media in order to promote the development of anaerobes. These substances, by considerably lowering the oxidation-reduction potential of the medium, instantly bring about the reduced state needed for the rapid start of a fermentation process or, in general, for the uninterrupted growth of anaerobes. The organism is commonly inoculated in a culture tube with a *deep* (10 mm) layer of nutrient broth, containing, for instance, thioglycollate and may then be incubated without further precautions "under aerobic conditions".

The use of ascorbic acid as a reducing agent for the same purpose was already recommended by ILLÉNYI and BÜSING (11) in 1939. In 1941 BÜSING (6) described the very marked improvement in the surface growth of *Clostridium tetani* on Zeissler blood-agar plates to which 0.1 % of sodium ascorbate had been added, when the Petri dishes were kept in an anaerobe jar (anaerostat). BÜSING also mentions incidentally, that he did not succeed in obtaining any growth of the organism on the same medium when the dishes were incubated "under external aerobic conditions".

It will become clear from what follows, that the addition of reducing substances to an agar-medium freely exposed to the air, is not only completely useless, but actually deleterious to any anaerobic or micro-aerophilic organism, on account of the formation of hydrogen-peroxide at the surface of the medium.

Hydrogen-peroxide, even in very small concentrations, is a powerful poison to the living cell and particularly to organisms which are devoid of catalase and unable to decompose H_2O_2 .

¹) The same result could be obtained, as was established later on, by the addition of gentian-violet 1:100,000 to the fermentation medium or to the nutrient agar. *Termobacterium*, a Gram-negative organism, was not influenced by the concentration of gentian-violet used, whereas the Gram-positive lactococci were completely inhibited.

It is interesting to recall to mind the fact, that one of the great pioneers in general microbiology, M. W. BEIJERINCK (1), was the first to demonstrate, already more than half a century ago (1893) and eight years before the name of catalase was given to this enzyme by LOEW (22), that the industrial lactic-acid bacteria do not decompose hydrogen-peroxide; "thus forming perhaps the only exception to the general rule, that living or dead protoplasm, which has not been heated to 60° C., always decomposes this substance". Other exceptions have been found since then; it has been shown that lactic-acid bacteria share the lack of catalase with a) obligate anaerobes, b) certain streptococci and sarcinae, e.g. *Pneumococci* and *Sarcina ventriculi* (BEIJERINCK (2)) and c) a few other microbes, such as *Shigella dysenteriae* (M'LEOD and GORDON, (24)) and *Acetobacter peroxydans* (VISSER 'T HOOFT (31)). According to M'LEOD and GORDON, a number of pathogenic organisms, like *Vibrio cholerae*, *Bact. typhosus* and *para-typhosus* A. *Bact. mallei*, and *Shigella paradysenteriae* show only slight catalase activity. Among the lactic-acid bacteria, dextranococci are not quite catalase-less (BEIJERINCK (3)); neither are propionic-acid bacteria (SHERMAN (30)).

As might be expected, the sensitiveness of the living cell to hydrogen-peroxide runs generally parallel to the presence or absence of catalase; the strict anaerobes being the most sensitive, directly followed by lactic-acid bacteria and many streptococci (24). The first are inhibited by $\pm 0.0003\%$ of H_2O_2 , whereas lactic-acid bacteria stand concentrations about ten times as high.

The formation of H_2O_2 in the course of oxidation processes is a very widespread phenomenon indeed; it is, in fact, the first reduction product of molecular oxygen, and appears with great regularity during the spontaneous, uncatalyzed oxidation of a great many substances: Nascent hydrogen, produced for instance by electrolysis of water, has been shown to give H_2O_2 by reacting with dissolved oxygen. SCHALES (27) by extremely sensitive methods has been able to show H_2O_2 formation during the oxidation of hydrogen-sulfide, sodium-hydrosulfite and hydrazine-sulfate. As to the appearance of H_2O_2 during the oxidation of organic substances, it will be sufficient to refer to the extensive work of WIELAND and FRANKE (32); the substance was demonstrated to occur during the autoxidation of leuco-methylene-blue, indigo-white, thiophenol, hydrazobenzene, pyrocatechol, pyrogallol and dihydroxy-maleic acid. SCHALES (27) demonstrated H_2O_2 formation during the autoxidation of cysteine, glutathione and thioglycollic acid; VON SZENT GYÖRGYI (30) found the same phenomenon with his "hexuronic-acid": the future ascorbic-acid. The detection of H_2O_2 during these autoxidations is made possible by the remarkable fact, that H_2O_2 itself reacts only slowly with several of the reducing substances (MACRAE (23)). This fact was shown by VON SZENT GYÖRGYI (30) to apply also to ascorbic acid, stable against H_2O_2 even

in the presence of the H_2O_2 -specific enzyme, peroxidase! This made it possible to use ascorbic acid in a chronometric method for the estimation of peroxidase-activity, worked out by the present author (8; 9).

Finally, hydrogen-peroxide may appear in the course of biochemical processes occurring in cell-free enzyme systems as well as within the cell. It is formed regularly in the course of enzyme-catalyzed dehydrogenations *in which free oxygen is capable to serve as a hydrogen-acceptor*. The enzymes concerned are the so called aerobic- or oxytropic- or aéro-dehydrogenases such as WARBURG's original yellow enzyme, SCHARDINGER's xanthine-oxidase, KREBS' d-amino-acid-oxidase and MÜLLER's glucose-oxidase. In fact, all these enzymes are now known to be "yellow enzymes", i.e. flavo-proteins (COULTHARD (7); VAN BRUGGEN (5)), characterized by their indifference towards H_2S , HCN , CO and other heavy-metal poisons. In living cells, therefore, the formation of H_2O_2 may be expected to occur in organisms showing some degree of flavoprotein respiration, but its presence can of course only be detected in organisms which are: a) devoid of catalase, so that H_2O_2 may accumulate, and b) potentially able to grow in an atmosphere containing free oxygen. For the latter reason, H_2O_2 production has never been demonstrated in strict anaerobes, whereas in aerobic cultures of pneumococci, some pathogenic streptococci, lactic-acid cocci and lactic-acid bacteria the formation of H_2O_2 has been demonstrated beyond doubt by several investigators (for which see OPPENHEIMER and STERN (26)) in different ways. A very ingenious method for the detection of the formation of H_2O_2 by some kinds of lactic-acid bacteria has recently been described by KNETEMAN (18). It is based on the specific resolving action of H_2O_2 on manganese dioxide under slightly acid conditions. The strong resolving power on Mn O_2 of ascorbic acid, even when neutralized and buffered, is significant in this respect and might also be explained by the generation of H_2O_2 during its spontaneous oxidation.

From the above it will be sufficiently clear, that the addition of some reducing agent to a thin layer of nutrient agar *freely exposed to the air* will result in an effect quite opposite to the purpose in view: the lowering of the oxidation-reduction potential. Such an addition is equivalent to a continuous supply of small quantities of H_2O_2 , generally slow enough to be destroyed during the development of organisms provided with a fair amount of catalase. If, however, the nutrient agar is set aside for some time before being inoculated, the amount of H_2O_2 accumulated may be bactericidal even to these organisms. This was indeed observed during the experiments with the mixture of *Termobacterium* and lactic-acid streptococci: after having been kept for some days in the ice-box, the medium had become unsuitable to *Termobacterium* as well as to the lactic-acid cocci.

These deductions are fully corroborated by the discovery of a *virucidal activity* of ascorbic acid reported to exist against different viruses *in vitro*, i.e. poliomyelitis-(JUNGBLUT (12)), vaccina-(KLIGLER and BERNKOPF (14))

and influenza A-virus KNIGHT and STANLEY (19)). This virucidal action was proved by KLEIN (13) to be equally due to the formation of H_2O_2 during the aerobic decomposition of ascorbic acid; it could be reproduced by the addition of corresponding quantities of H_2O_2 and it could be annihilated by rabbit-blood catalase.

In the preceding pages mention has been made of the generation of H_2O_2 by several flavoproteins during their enzymatic activity, i.e. *in the presence of their specific substrate and of oxygen*. Consequently, the addition of d-amino-acids together with d-amino-acid oxidase to the medium should prove to be bactericidal, and so should MÜLLER's glucose-oxidase in the presence of glucose, etc.

In fact, this last case has already been met with during investigations on the penicillin-production of *Penicillium notatum*. From some cultures of this mould a filtrate was obtained, which showed bactericidal action against *Staphylococcus aureus*, (a catalase-positive aerobe!) only in the presence of glucose; at the same time gluconic acid was formed. The bactericidal product was first called Penicillin A to distinguish it from "real" Penicillin; later on it was named Notatin to avoid confusion (COULTHARD (7)), and very probably the same substance was examined and called Penicillin B by another group of investigators (VAN BRUGGEN et al (5)).

When purified, this Notatin proved to be an aero-dehydrogenase belonging to the flavoproteins and at least very similar to the glucose-oxidase, which MÜLLER (25) had found in the mycelium of *Aspergillus niger* and "*Penicillium glaucum*" (whatever the species designated by this name may have been!)

Finally SCHARDINGER's xanthine-oxidase, the flavoprotein of milk, catalyzing the oxidation of hypoxanthine to uric acid in the presence of air, has been shown by GREEN and PAULI (10) to possess a bactericidal activity analogous to that of notatin; once more the bactericidal action could be traced down to the H_2O_2 , appearing during the activity of the enzyme in the presence of oxygen.

Summary.

The selective inhibition of lactic-acid bacteria, when sown on nutrient agar to which 0.1 % of sodium ascorbate has been added, is caused by hydrogen-peroxide, generated during autoxidation of the ascorbic acid. Hydrogen-peroxide is a powerful bactericide for all kinds of micro-organisms; especially catalase-negative microbes, i.e. strict anaerobes and the facultative aerobes constituting the group of lactic-acid ferments are very sensitive to H_2O_2 ; they are inhibited by concentrations of the order of 0.001 % ($1 : 10^5$). This concentration will be easily attained by the presence of reducing substances, which, being liable to autoxidation in the presence of air, will give rise to H_2O_2 , which accumulates in the nutrient agar. This H_2O_2 will first of all inhibit catalase-negative organisms able to grow

in the presence of oxygen, i.e. lactic-acid bacteria. When the dishes are stored some time before inoculation, the amount of H_2O_2 generated may also become harmful to catalase-positive organisms.

Thus, lactic acid ferments, which are easily obtainable, are exceptionally suitable to demonstrate the H_2O_2 -type of bactericidal action. The bactericidal action of flavoproteins in the presence of their specific substrates and of oxygen is also known to be due to H_2O_2 formation. In the same way virucidal action *in vitro* by ascorbic acid is proved to be a H_2O_2 poisoning of the virus.

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Anatomy. — *Occurrence of mitotic divisions in glomeruli in the kidney of pregnancy.* By PH. H. HARTZ. (Curaçao N.W.I.)

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Although it has been known since many years that the significant lesions of "the kidney of pregnancy", i.e. eclampsia and prae eclampsia, are to be found in the glomeruli, there still exists much difference of opinion with respect to their morphology and nature. TH. FAHR (as quoted by BELL (1) and MASTBOOM (2)) found that in eclampsia the glomeruli were enlarged and ischaemic. The walls of the capillaries were swollen, the cell borders were ill-defined and sometimes the nuclei seemingly had disappeared. Only exceptionally was an increase of endothelial nuclei observed, which FAHR attributed to an associated infection. FAHR considered the lesions as degenerative and for this reason used the term glomerulonephrosis as designation. It should be noted that FAHR did not use modern methods for the staining of the basement membranes of the glomerular capillaries.

BAIRD and SHAW-DUNN (3) and KELLAR, ARNOTT and MATHEW (4), using the azocarmine-aniline blue stain, found in eclampsia and prae eclampsia ischaemia and a moderate enlargement of the glomeruli, swelling of the endothelial cells and sometimes an increase of endothelial nuclei. There was also swelling and sometimes fibrillation of the capillary basement membranes. The authors considered the lesions as inflammatory. BELL (1, 5) using the same technic, found in fatal cases of eclampsia a marked narrowing of the glomerular capillaries, caused usually by an increase in thickness of the capillary basement membranes but sometimes by a increase of endothelial cells. Some increase of endothelial cells was found in many cases and occasionally this increase was rather striking but never pronounced enough to warrant a diagnosis of acute glomerulonephritis. Originally BELL (1) was inclined to consider the increase of endothelial cells as an inflammatory phenomenon, that is, a form of glomerulonephritis. He also stressed its frequent occurrence. Later on he changed his opinion and stressed the degenerative nature of the glomerular lesions (5).

Recently MC. MANUS (6) investigated the glomerular lesions in 16 cases of eclampsia with his periodic acid-Schiff's reagent technic (7, 8) and came to conclusions which are fundamentally different. In contrast with the authors cited before he did not find thickening of the basement membranes of the glomerular capillaries or proliferation of the endothelial cells but constant swelling and vacuolation of the cells of the mesangium

or axial spaces (9, 10) which sometimes also involved the endothelial cells. MC. MANUS considers this vacuolation or "reticulation" as the specific lesion of the glomerulus in eclampsia.

It follows from this short review of the literature that one of the disputed points is whether proliferation-especially of the endothelial cells-really does occur in the glomeruli of the kidney of pregnancy. Proliferation on a restricted scale can only be proved with certainty by the finding of mitotic divisions in the cells concerned and for this the material of the authors cited was unsuited since it was obtained from routine autopsies and, as BAIRD and SHAW DUNN (3) and KELLAR and associates (4) specially mention, already showed autolysis. As I had already been able to demonstrate mitoses in glomeruli in glomerulonephritis, malignant sclerosis and septicaemia (11) and later in cases of pyelonephritis, pneumonia and subacute bacterial endocarditis (12) in all of which autopsy had been performed within 45 minutes after death, it was deemed worth while to search for mitoses in the glomeruli of the kidney of pregnancy in such cases in which renal tissue could be obtained under the same favorable circumstances.

Material and Methods.

In case No. 1 autopsy was performed 30 minutes after death, in cases Nos. 2, 3 and 4 not more than 5 minutes after death. Blocks of renal tissue, not more than 3 mm thick, were cut for fixation. In case No. 1 the Bouin-sublimate mixture was used as fixative; in the other cases the tissues were fixed in Stieve's fluid, which in cases Nos. 2 and 4 was modified by replacing the glacial acetic acid by a 2 per cent aqueous solution of trichloroacetic acid (13). The blocks of tissue were embedded in paraffin by Péterfi's methylbenzoate-celloidin method and the sections stained with the periodic acid-Schiff's reagent technic, with Masson's tetrachrome, with the azan stain and with orcein.

Case 1.

A coloured woman, 32 years old, multigravida, was admitted in the hospital with the classic symptoms of eclampsia and died after a few hours. As the patient had only very recently immigrated from one of the British West Indian islands and moreover was not married, no anamnestic data were available.

Gross notes. The liver weighed 2000 grams and showed the haemorrhagic necroses typical for eclampsia and an enormous subcapsular haematoma. The kidneys weighed 300 gram; in the cortex there were numerous small cysts, on the cut surface the markings were normal. The heart weighed 385 gram; there were many subendothelial haemorrhages in the left ventricle, especially in the ventricular septum. The uterus contained a dead, almost full term child.

Microscopic examination of the kidneys.

The glomeruli varied in size; sometimes they appeared to be smaller than normal so that the capsular space was fairly wide. Other glomeruli were somewhat enlarged. The epithelial cells were distinct. The capillary basement membranes were very little thickened. The axial spaces were very distinct and the cells contained in them were conspicuous and sometimes vacuolated. The capillaries contained very little blood; occasionally there were a few leukocytes. In many glomeruli there was an increase of endothelial cells in several but not all of the loops.

In a few glomeruli mitoses were found. All stages of the mitotic division were represented; centrioles and achromatic spindles were very distinct. In some instances they occurred probably in the cells of the axial spaces, but others could with certainty be localized inside the capillaries. No mitoses were found in the epithelial cells. The lumina of the proximal convoluted tubules were wide and contained a little precipitate. Their epithelium was somewhat swollen and sometimes contained small vacuoles. Necrosis or hyaline droplet degeneration were completely absent. A few loops of Henle and distal convoluted tubuli contained hyaline casts.

The small cortical arteries and arterioles were normal; only in the larger arteries a slight thickening of the intima was found.

Case 2.

A coloured woman, 32 years old and primigravida, was admitted into the hospital with typical eclampsia and died 18 hours after the onset of the symptoms. One week before being admitted her urine had been examined and was found to be normal.

Gross notes. The liver weighed 1600 gm and showed the typical haemorrhagic necroses. The kidneys weighed 327 gm, their color was brown with a few greyish spots. On section the markings were not distinct. The pyelum of the right kidney was slightly distended. The heart weighed 250 gm and did not show anything abnormal.

The uterus contained a dead child the size of which corresponded with the ninth month of pregnancy.

Microscopic examination of the kidneys.

Only part of the glomeruli were moderately enlarged but many showed an increased cellularity. The epithelial cells could easily be recognized by their localization and their typical nuclei. The basement membranes of the glomerular capillaries were thin; the axial spaces were broadened and contained very conspicuous, sometimes large and finely vacuolated cells. The capillaries contained very little blood; in many loops the endothelial cells were swollen and in some places seemed to block the lumen. The protoplasm of these swollen epithelial cells was often vacuolated. In still other places isolated glomerular loops were blocked by

swollen endothelial cells and by very thin fibers which stained red with the periodic acid-Schiff's reagent technic. In a few glomeruli mitoses were found. They occurred in the cells of the axial spaces and in the endothelium. No mitoses were found in the glomerular epithelium.

The lumina of the proximal convoluted tubules were wide; they contained only little precipitate. Their epithelium was of normal height; in many cells the rods were distinct, in others they were replaced by small oxyphilic granules arranged in rows. In the distal convoluted tubules and in the collecting tubules there were acidophilic casts and large accumulations of a strongly acidophilic granular precipitate. Sometimes a cast in a distal convoluted tubule was covered by flattened epithelial cells. In many collecting tubules containing casts or precipitate there was patchy necrosis and desquamation of the epithelium.

The smaller cortical arteries and the arterioles were normal. A few larger arteries showed some thickening of the intima.

Case 3.

A coloured woman, 17 years old and primigravida, mens. IX entered the hospital with the typical symptoms of eclampsia and died after a few hours. No anamnestic or clinical data were available. At autopsy haemorrhagic necroses were found in the liver.

Microscopic examination of the kidneys.

Part of the glomeruli were moderately enlarged and there was sometimes a definitely increased cellularity. The epithelial cells were larger than normal; sometimes contained small hyaline droplets. The basement membranes of the capillaries appeared as thin, sometimes wrinkled lines. The axial spaces contained large connective tissue cells with finely vacuolated protoplasm. The capillaries contained very little blood, many loops were plugged by swollen endothelial cells also with vacuolated protoplasm. In other loops the endothelial cells were smaller but the lumina were compressed by the broadened axial spaces. Sometimes the peripheral part of a loop consisted of an inextricable mass of cells and very thin fibers.

Mitoses were found in both the cells of the axial spaces and in the endothelium. Sometimes 2 mitoses were observed in the section of one glomerulus.

There was some slight swelling of the epithelium of the proximal convoluted tubules. The bloodvessels did not show pathologic changes.

Case 4.

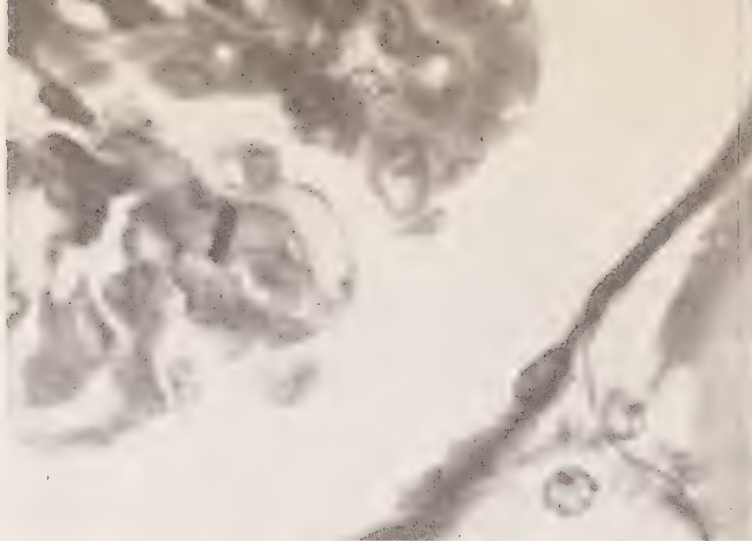
A coloured woman, 36 years old and VII gravida, entered the hospital because of albuminuria and gross edema of the legs. At the end of every previous pregnancy she had suffered from albuminuria, hypertension and edema but had always completely recovered after delivery. On admission the blood pressure was 230/140. One day after admission she became comatose and died suddenly.



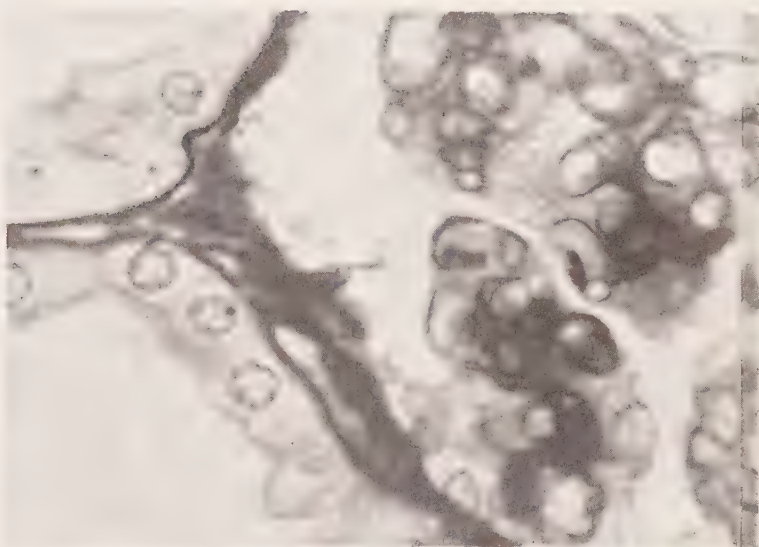
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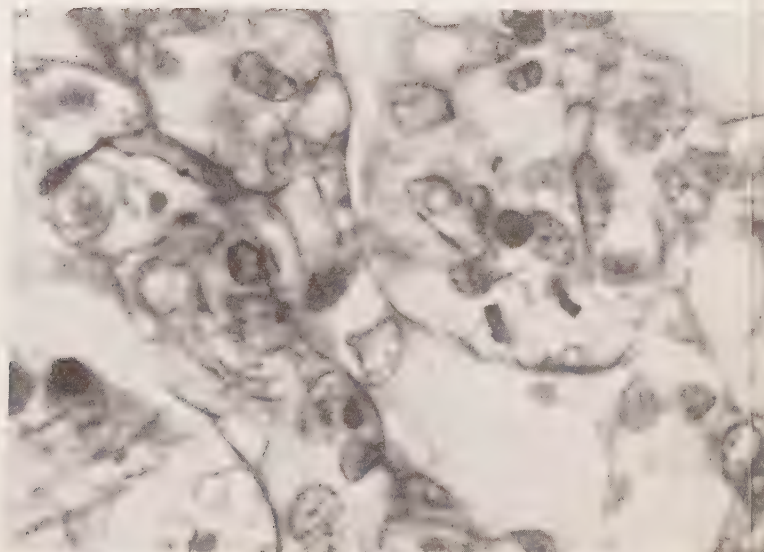
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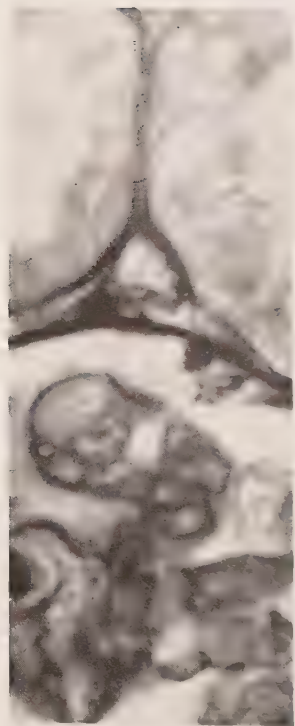
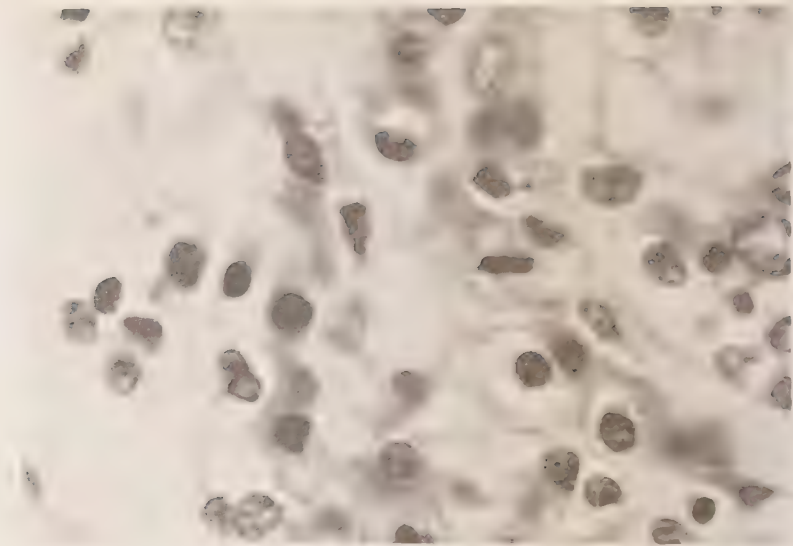
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PHOTOMICROGRAPHS.

- 1. Case 1. Mitosis in endothelial cell. Periodic acid-Schiff's reagent technic.
- 2. Case 1. Mitosis in cell in axial space. Masson's tetrachrome with Fastgreen F. C. F.
- 3. Case 2. Mitosis in cell in axial space. Periodic acid-Schiff's reagent technic.
- 4. Case 3. Mitosis in endothelial cell. Periodic acid-Schiff's reagent technic.
- 5. Case 4. Mitosis in endothelial cell. There is some fibrillation of the basement membrane. Periodic acid-Schiff's reagent technic.
- 6. Case 4. Mitosis in cell in the axial space. Masson's tetrachrome with Fast Green F. C. F.
- 7. Case 3. Swollen and finely vacuolated endothelial cell. Periodic acid-Schiff's reagent technic.

Gross notes. The kidneys weighed 422 gm, the cortex was swollen and pale. The pyelum of the right kidney was slightly distended. The liver weighed 1940 gm and did not show necroses or haemorrhages. The heart was hypertrophic and weighed 404 gm.

A living, almost full term child, which died shortly afterwards, was removed from the uterus at the beginning of the autopsy.

Microscopic examination of the kidneys.

The majority of the glomeruli were enlarged; there was also increased glomerular cellularity. The glomeruli contained very little blood and an occasional glomerulus appeared to be solid. A few epithelial cells showed hyaline droplet degeneration. There was some "reticulation" of the axial spaces. The capillary basement membranes were thin; the number of endothelial cells of many loops was definitely increased and many endothelial cells were swollen. There were very few glomeruli which showed some shrinking combined with a thickening of the capillary basement membranes.

Mitoses in the glomeruli were found fairly easily in the endothelium as well in the cells of the axial spaces; sometimes 2 mitoses were observed in the section of one glomerulus. They were found in all stages. The tubuli were wide and contained a little granular precipitate. There was no degeneration of the tubular epithelium. A few mitoses were observed in the epithelium of the proximal convoluted tubuli.

Hyaline degeneration of the arterioles was completely absent; in the larger arteries there was sometimes a slight thickening of the intima and occasionally a duplication of the inner elastic membrane.

Discussion.

In the first 3 cases the diagnosis "eclampsia" was obvious; in case no. 2 there existed probably haemoglobinuria and microscopically lesions usually present in "lower nephron nephrosis", but here on a restricted scale, were found in addition to the glomerular lesions; a combination which is not infrequent in eclampsia (14). In case no. 4 the clinical diagnosis was prae-eclampsia; the fact that apart from slight changes in the larger arteries and a few shrunken glomeruli the renal lesions were identical with those found in the other 3 cases and that, apart from the hypertrophic heart no other pathologic changes were found at autopsy, makes the diagnosis reasonably certain.

From the microscopic examination several facts emerge. In accordance with the findings of BELL (1, 5) and of BAIRD and SHAW DUNN (3) the tubuli showed very little pathologic changes. It is therefore very probable that the severe degenerative lesions observed by FAHR were, in so far as he was not dealing with cases of beginning cortical necrosis, caused by autolysis.

The changes observed in the glomeruli consisted of the vacuolation of the axial spaces, first described by MC. MANUS, occasional fibrillation of the capillary basement membranes, swelling of the endothelial cells and the presence of mitotic divisions in both the cells of the axial spaces and in the endothelial cells. Thickening of the capillary basement membranes, when present, was discrete. There were no mitoses in the glomerular epithelial cells.

The finding of mitotic divisions in glomeruli in the kidney of pregnancy has never been reported before. Since mitoses are irrefutable proof of proliferation the significance of proliferation in glomeruli has to be considered. A review of modern textbooks and monographs (5, 15, 16, 17, 18, 19) shows that proliferative manifestations are considered the most important and prominent feature of inflammation of the glomerulus in clinical acute glomerulonephritis and in the subclinical glomerulonephritis often found in various infections. The proliferation of the endothelial cells is the most striking, but according to MC. MANUS some of the additional cells are to be found in the axial spaces (6). Other manifestations of the inflammatory reaction in clinical acute glomerulonephritis consist of swelling of the endothelial and sometimes of the glomerular epithelial cells, of discrete thickening of the capillary basement membranes with, in later stages, the formation of intracapillary fibers, whereas exsudation may be quite inconspicuous or almost completely absent. Epithelial crescents are only found in combination with haemorrhages (18). In our cases all these features, even the formation of intracapillary fibers in an isolated capillary loop, were present, though less strikingly so than in clinical glomerulonephritis. It is therefore tempting to consider the glomerular changes in the kidney of pregnancy as inflammatory. Against this conception two objections can be advanced: 1. the constant presence of vacuolation of the axial spaces in the kidney of pregnancy, which so far has not been found in glomerulonephritis and 2. that in my cases there was glomerulonephritis as complication, a suggestion already made by FAHR. Against the first objection it can be argued that, as the inflammatory reactions in glomerulonephritis and in eclampsia and prae-eclampsia are probably due to different causative factors, morphologic differences are to be expected as happens also in inflammation caused by different infective agents. That nephritis, which in Curacao is very rare, should in the absence of infectious processes complicate every one of the reported cases is so improbable, that further discussion is superfluous.

The conception of the glomerular lesions in the kidney of pregnancy as inflammatory in nature is certainly not new, but is rendered much more probable by the proof that proliferation really occurs. There is certainly no reason for describing the glomerular lesions as purely degenerative and since, as has already been mentioned, the tubuli show very little pathologic changes, FAHR's term "glomerulonephrosis", which still figures in several textbooks, should be discarded.

Summary.

In four cases of the "kidney of pregnancy" in which autopsy could be performed in very favorable circumstances for the subsequent microscopic examination, mitotic divisions were found in the glomeruli. They occurred in the endothelial cells of the capillaries and in the cells of the axial spaces. Since endothelial proliferation in glomeruli is considered the most important feature of glomerular inflammation and other lesions, usually found in glomerulitis, were also present, it is suggested that the glomerular lesions of the kidney of pregnancy are inflammatory in nature.

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Chemistry. — *Influence of organic compounds on soap and phosphatide coacervates — IX*¹⁾ — *Notes on the action of some organic ring systems with and without polar groups on soap coacervates.* By H. L. BOOIJ. (Communicated by Prof. H. G. BUNGENBERG DE JONG.)

(Communicated at the meeting of November 26, 1949.)

1. Introduction.

The influence of many organic compounds with a polar group on the oleate coacervate has already been measured by BUNGENBERG DE JONG, SAUBERT and BOOIJ (1937, '38) and ROSENTHAL (1939). These experiments were explained by assuming that such molecules exert two opposite forces: a "condensing" action of the hydrocarbon chains and an "opening" action of the polar groups. The picture, however, must be considerably more complex as KOETS and BUNGENBERG DE JONG (1938) showed that aliphatic hydrocarbons do not fit into this picture (for instance *n*-octane having a condensing action at low and an opening action at somewhat higher concentrations). The aromatic hydrocarbons did not show this anomalous behaviour.

BOOIJ and VELDSTRA (1949) investigated the influence of various hydrogenated derivatives of naphthalene acetic acid on an oleate coacervate at high p_H and found that decahydronaphthalene acetate had a much stronger (opening) action than less hydrogenated derivatives of naphthalene acetate. This might be explained by assuming that the decalin nucleus shows a stronger affinity to the apolar part of the oleate micelle than the naphthalene nucleus. The action of these relatively small ions is for a large part determined by the amount of these anions taken up in the soap micelles (cf. the difference between nonylate and undecylate; BOOIJ and BUNGENBERG DE JONG showed that a large percentage of nonylate added to an oleate coacervate remained in the watery medium, whereas undecylate is taken up strongly in the micelles). In the case of decahydronaphthalene acetate (stronger affinity to the non-polar part of the micelles) a greater uptake in the micelles would take place as compared with naphthalene acetate, resulting in a stronger action. This difference led us to the idea to compare the influence of naphthalene and decalin (which, however, proved to be reverse; see section 2). After that it seemed worth while to investigate the action of other nuclei (especially heterocyclic ones).

¹⁾ This series was published under the name: "Einfluss organischer Nichteletrolyte auf Oleat- und Phosphatidkoazervate" in *Protoplasma* 28, 498, 543 (1937); 29, 481, 536; 30, 1, 53, 206 (1938), and under the name: "Influence of organic compounds on oleate and phosphatide coacervates" in *Bioch. Bioph. Acta* 3, 242 (1949).

2. *On the action of some aromatic, alicyclic and heterocyclic nuclei.*

To 70 ml oleate solution (10 g Na-oleate + 525 ml H₂O + 25 ml KOH 5 mol) the desired quantity of the organic substance is added, after which the solution is gently rolled in order to dissolve the added compound. Sometimes this will take a whole day. Then varying amounts of KCl were added according to the following scheme:

$$\left\{ \begin{array}{l} 5 \text{ ml oleate solution} \\ x \text{ ml KCl } 3.8 \text{ n} \\ (14 - x) \text{ ml H}_2\text{O.} \end{array} \right.$$

The mixtures were thoroughly shaken and set in a thermostate of 25° C. Next day the volumes of the coacervates formed were read and plotted against the KCl concentration (see for the method also BOOIJ and BUNGENBERG DE JONG, 1949). Organic substances shift this KCl-curve and this shift — expressed in KCl concentration (mol/l) — is taken as a measure of the action of the substance added. The shift may go to lower KCl concentrations (—; "condensing action") or to higher (+; "opening action").

The condensing action of decalin appears to be appreciably smaller than that of naphthalene and tetralin (fig. 1). One would not expect this

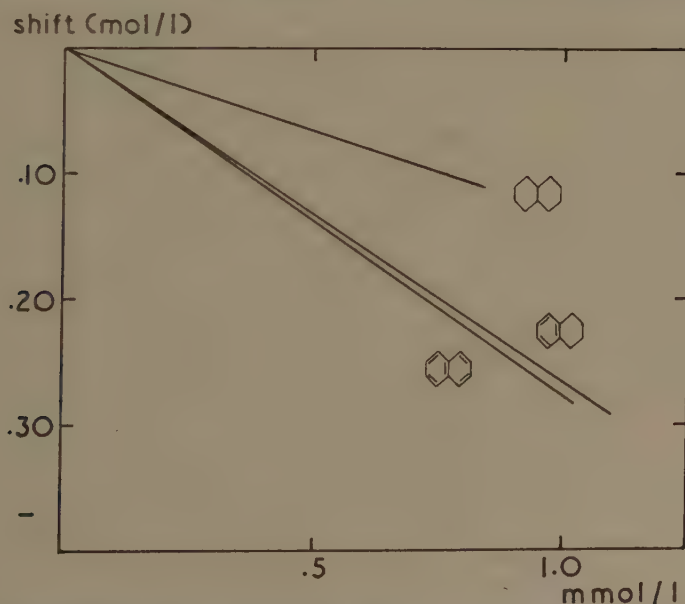


Fig. 1. Condensing action of naphthalene, tetralin and decalin on an oleate coacervate. Absc = concentration of added substance, ordinate = shift of the coacervation curve expressed in KCl concentration.

after the experiments of BOOIJ and VELDSTRA (1949). They explained the difference between decahydronaphthalene acetate and naphthalene-acetate by supposing that the former compound is taken up more readily by

the micelle as the affinity of the decalin nucleus to the apolar part of the oleate micelle is somewhat greater than that of the naphthalene ring system. In the case of naphthalene and decalin, however, one may safely assume that both substances are taken up to a large extent into the micelles and it is not yet clear why the situation is exactly the reverse of that with the acetates.

With heterocyclic ring systems the results are easily explained. One would suppose that the exchange of a carbon atom of an aromatic ring for another atom will result in a decrease of the action. This decrease will be greatest with N, less with O and still less with S. Generally speaking the aromatic character of the resulting heterocyclic ring grows stronger in this direction. Especially the NH group will give the compound a more polar character (for instance furane and thiophene resemble benzene, while pyrrol is to be compared with phenol). Fig. 2 and 3 demonstrate this clearly. Introduction of a N-atom into a ring results in a marked decrease of activity (cf. indole — indene; quinoline and isoquinoline — naphthalene).

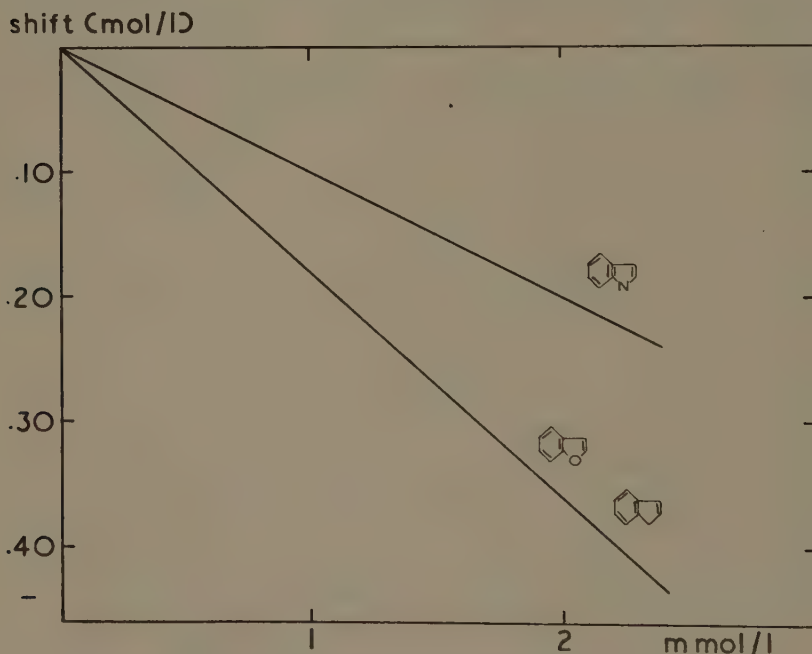


Fig. 2. Influence of heterocyclic ring systems (coumarone and indole) on an oleate coacervate as compared with that of indene.

This decrease is much smaller or absent with the O-atom (cf. coumarone-indene). We got the impression that furane has a smaller activity than thiophene, but we could not prove that beyond doubt. The difference between acridine and diphenyleneoxyde was easily demonstrated. Pyridine and piperidine distinguish themselves markedly (fig. 4). If it may be assumed that at high p_H (we worked at a p_H 12–13) the polar groups

are not dissociated, the most important difference of these molecules would be their aromatic or alicyclic character. Then here too the substance with the alicyclic ring has the strongest action. This would lead to the idea that

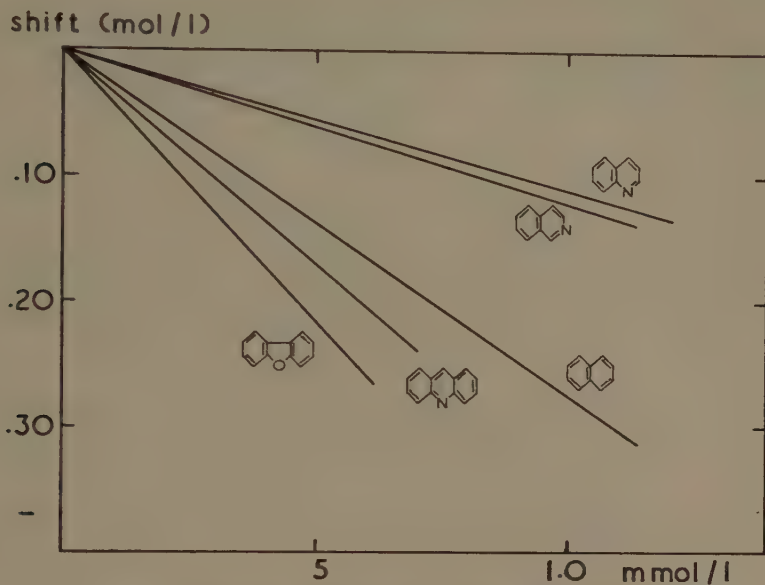


Fig. 3. The action of quinoline, isoquinoline, acridine and diphenyleneoxyde on the oleate coacervate as compared with naphthalene.

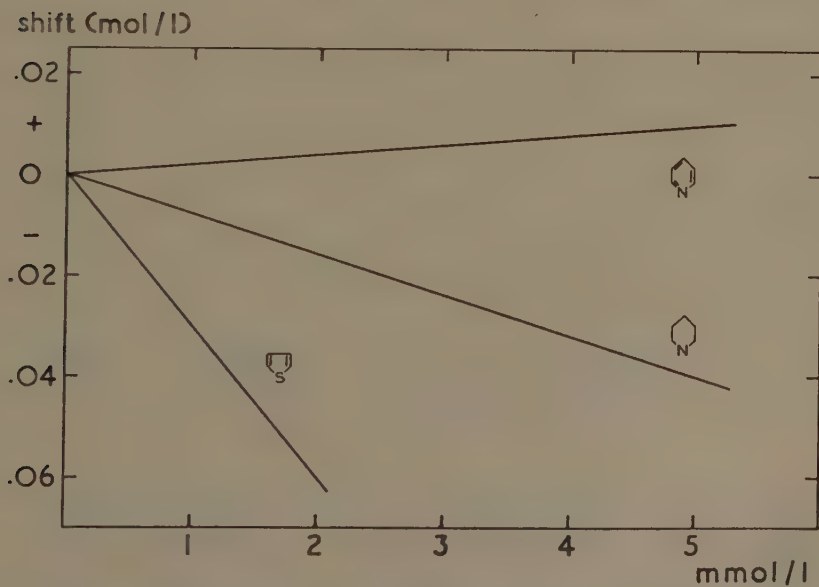


Fig. 4. Influence of pyridine as compared with piperidine and of thiophene on the oleate coacervate. Note that the action of these small molecules is much less than that of bigger molecules. (An opening action (pyridine) is expressed by the sign +, a condensing action by —.)

a substance with an alicyclic ring has the strongest action provided a polar group is part of the molecule. When the compounds have no polar groups (fig. 1) the reverse will be true.

Finally we found that imidazole, piperazine and pyrazole had no influence on the coacervate in the concentrations measured (up to 1 mmol). Here it may be supposed that the substance added will be found practically exclusively in the watery medium, so that the action on the micelles is very slight.

3. The influence of introduction of a polar group.

First we measured the action of the isomeric methylcyclohexanols as compared with cyclohexanol and cyclohexanon (fig. 5). The place of the methylgroup influences the action, the longest molecule having the greatest

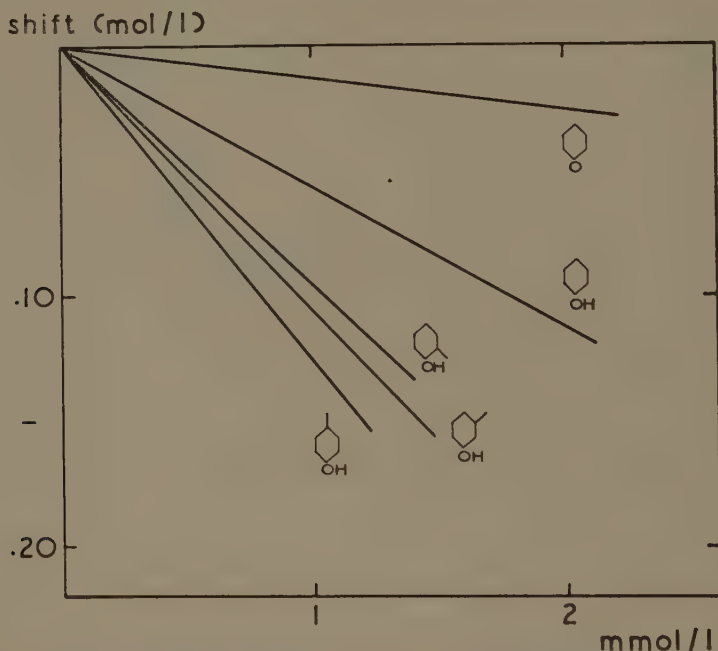


Fig. 5. The action of ring systems with a polar group depends on the nature of the polar group (cyclohexanon < cyclohexanol), the hydrophilic-hydrophobic balance (cyclohexanol < methylcyclohexanol) and the length of the molecule (see the difference between the isomeric methyl cyclohexanols).

activity. It had been found already that the keton-group diminishes the "condensing" action if introduced in stead of an OH-group (BUNGENBERG DE JONG et al., 1938).

These results were in accordance with former publications in this field. An investigation of terpenes and related substances showed a somewhat surprising phenomenon (fig. 6). This fact was the relatively low condensing activity of pinene and d-limonene. A priori one would not expect that

menthol, borneol and terpineol have a stronger action than the apolar substances mentioned. If we would try to explain these facts along the lines

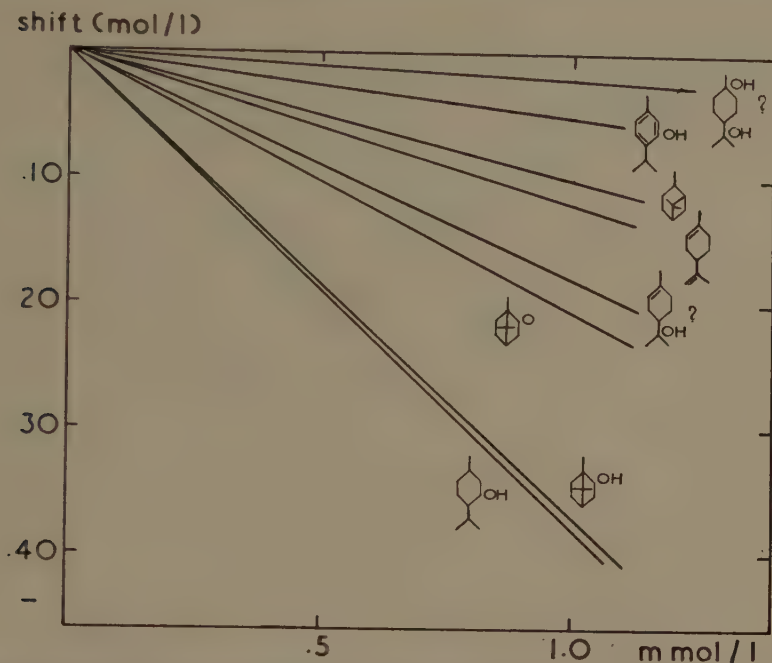


Fig. 6. The action of pinene, *d*-limonene, camphor, borneol, menthol, thymol, terpineol (mixture of isomers) and terpinhydrate (mixture of isomers) on an oleate coacervate.

formerly developed, it would be necessary to assume that a double bond exerts a stronger opening action than an OH-group, which is contrary to many known facts. So it seems evident that another factor plays a part here.

Many facts have been found to be exactly as expected. Obviously borneol has a much stronger activity than camphor (difference of alcohol and keton-group). Introduction of a second OH diminishes the action very markedly (see terpinhydrate). The difference between thymol and menthol must be ascribed to the high p_H at which these experiments were performed. Then the aromatic OH-group will be present in the dissociated phenolate form, which must result in a much smaller action.

4. Experiments on a coacervate of cetylsulfate.

It might be possible that the difference in action of naphtalene and decalin (see fig. 1) is based on a strong affinity of the aromatic nucleus to the double bond of the oleate molecule. Then a coacervate of a soap with a saturated carbon chain should show the reverse phenomenon. Obviously, this explanation would not help us to see why an alicyclic nucleus with a polar group does act more strongly than a similar aromatic derivative. An

experiment with a coacervate of cetylsulfate showed that the phenomenon is in that case exactly the same (fig. 7).

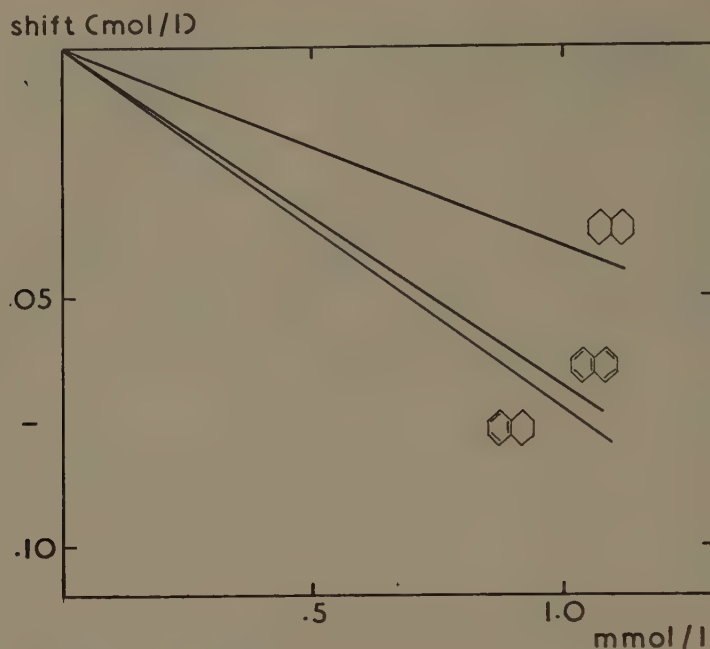


Fig. 7. The influence of naphthalene etc. on a coacervate of cetylsulfate is comparable to that on an oleate coacervate. Here the activity is expressed in the shift of the NH_4Cl curve.

One must work at a high temperature to coacervate a solution of cetylsulfate. The method we used was as follows: 5 g Na-cetylsulfate was dissolved in 200 ml distilled water and kept at a temperature above 60°C , as otherwise the soap will crystallize. Here too, as in the case of our experiments with oleate, a certain amount of the substance to be measured is added to a known volume of the soap solution (20 ml) and dissolved. Then we make the following mixtures:

$$\left\{ \begin{array}{l} x \text{ ml } \text{NH}_4\text{Cl } 5 \text{ n} \\ (7 - x) \text{ ml } \text{H}_2\text{O} \\ 3 \text{ ml cetylsulfate solution.} \end{array} \right.$$

The cetylsulfate is added to the salt-water mixture when the latter has reached the temperature of the thermostat in which the experiments are performed (66°C). These mixtures are well shaken and because of the high temperature heavy test tubes with ground glass stoppers have been used. Next day the volumes of the coacervates formed are read and plotted against the concentration of NH_4Cl . As usually, the shift of the coacervation curve under the influence of the added substance is taken as a measure of the activity of that substance.

Fig. 7 shows that the relation between naphthalene and decalin is exactly as with the oleate coacervate. This means that the double bond in the oleate molecule does not play an important part as regards this phenomenon.

We had the opportunity to investigate with this coacervate some other questions which arose during our experiments with oleate coacervates. The relation between borneol and camphor (fig. 8) is comparable with that on

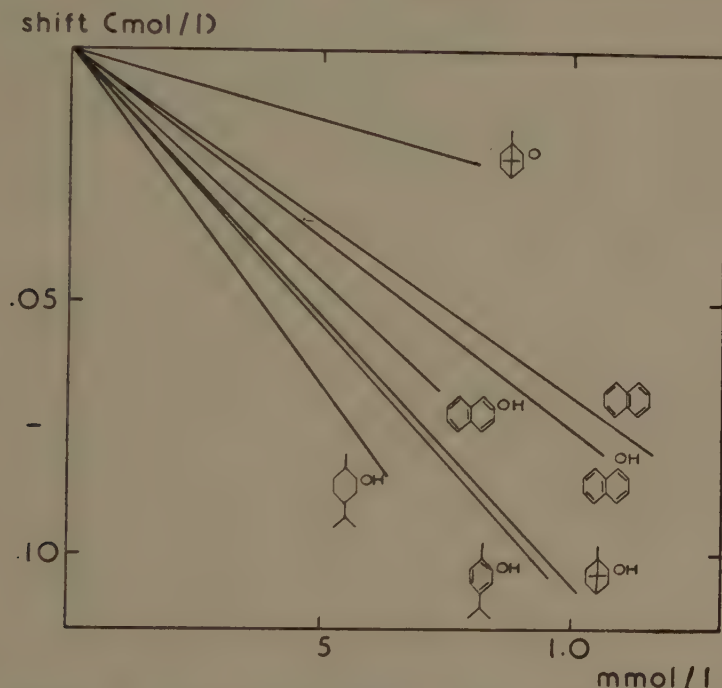


Fig. 8. Activity of α and β naphthol, naphthalene, camphor, borneol, menthol and thymol with a cetylsulfate coacervate as substratum. Note the much more pronounced activity of thymol as compared with that on oleate (fig. 6).

the oleate coacervate (fig. 6). Here too the keton group diminishes the condensing activity. As in this coacervate the p_H is in the neighbourhood of the neutral point (NH_4Cl was used as salt, while no KOH was added to the soap) one would expect that thymol and menthol will not show such diverging actions as with oleate (cf. fig. 8 and 6). This proved to be the case, menthol having a slightly stronger action, in agreement with the rule that an alicyclic compound with a polar group has a stronger action than the related aromatic one. Experiments on the naphthols gave us one more example of substances with a polar group with a stronger action than the parent substance (cf. the examples menthol, terpineol etc. already mentioned; there it was not possible to compare them with their real parent substances). Of course, β naphthol is somewhat more active than α naphthol, as its molecule is somewhat longer (see also BOOIJ and VELDSTRA 1949,

page 265 for the difference between naphthalene (1) and naphthalene (2) derivatives).

5. Discussion.

Two main points may be derived from our experiments:

a) The difference between alicyclic and aromatic nuclei is rather complicated, as the action depends on the fact whether a polar group is bound to the molecule. Without polar group the aromatic nucleus has a stronger action than the similar alicyclic ring, while with a polar group the alicyclic derivative is the stronger one. The phenomenon that an alicyclic ring without a polar group has a lower condensing action than the comparable aromatic one seems to be a general rule. For example, *o*-xytol and dimethylcyclohexane behave in the same manner (fig. 9).

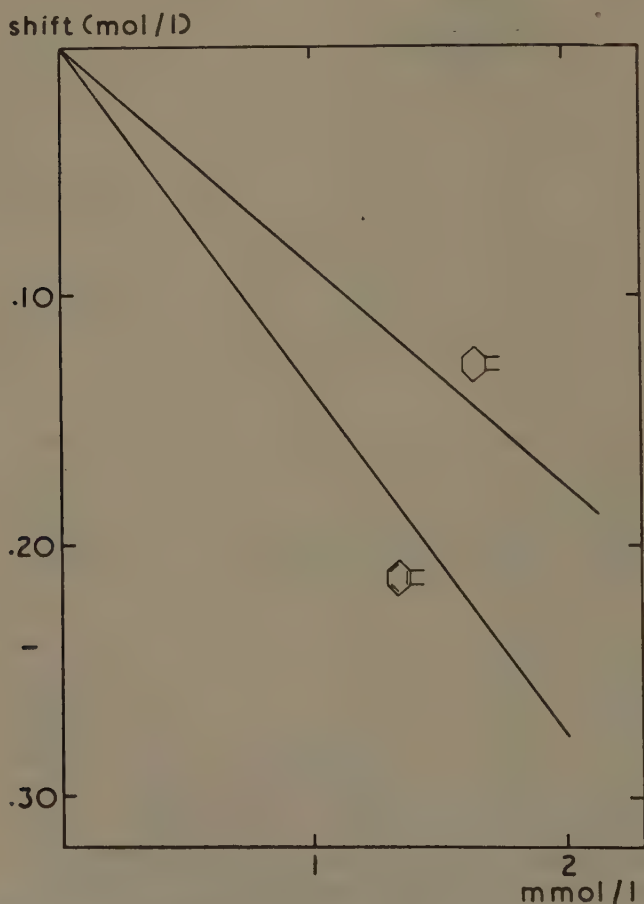


Fig. 9. The action of dimethylcyclohexanol on oleate as compared with that of *o*-xytol.

b) Some substances with a polar group exert a stronger action than their parent substances (cf. naphthol and naphthalene), a fact which is not in accordance with rules formerly laid down.

Several facts led to the conclusion (BOOIJ, 1949) that organic molecules may act in two ways on an oleate micelle. A substance like benzene might act a) in between the parallel carbon chains of the soap molecules and b) it might form a layer between the CH_3 end groups of these molecules resulting eventually in the well known „sandwich” micelle of Hess et al. (1941).

When we ask ourselves what molecular properties would give a preference to action a or b, we may start from the action of fatty anions on the oleate coacervate (BOOIJ and BUNGENBERG DE JONG, 1949). This action can only be explained by assuming that the fatty anions added will be arranged parallel to the carbon chains. This seems logical as strong forces will oppose the introduction of the charged polar group into the apolar inside of the micelles. An investigation of the homologous series of normal alcohols ²⁾ led to the same conclusion as regards the OH group. Obviously, in these cases the polar group serves as a kind of anchor, holding the molecule in the position between the parallel carbon chains of the soap molecules.

The logical consequence is that without that anchor the added substance may be found between the planes of the CH_3 -groups. So hydrocarbons might be expected to concentrate in the latter place. If we now see that n-octane (KOETS and BUNGENBERG DE JONG, 1938) shows at low concentrations a condensing and at higher ones an opening action, then we might assume that the concentration of n-octane within the inside of the micelle results in an “opening” action. The mechanism of this action (which must be distinguished from the opening action of fatty anions) is far from being understood.

As a working hypothesis we may adopt, however, that the action of hydrocarbons depends on their distribution within the micelle, two places (between the carbon chains and between the CH_3 -planes) being available. Analogies with the action of compounds with a polar group then teach us that concentration at the former place gives a “condensing” action ³⁾ while the added substance in the latter place acts as an “opening” compound. In the light of this hypothesis the behaviour of alicyclic and aromatic ring systems loses its enigmatic character.

The aromatic nuclei may be seen as somewhat more “polar” than similar alicyclic rings. Thus the alicyclic rings (without polar groups) tend to concentrate more in the inside of the micelles, in other words they have a less pronounced “condensing” action than the aromatic rings. When, however, they are provided with a polar group like OH etc. this tendency is annihilated and the stronger affinity of the alicyclic ring to the apolar chains of the oleate comes to the fore. Here the “condensing” action will be stronger than with comparable aromatic ring systems.

Our working hypothesis thus throws light on the two points mentioned

²⁾ Not yet published.

³⁾ With the exception of long fatty anions.

at the beginning of this section. The difference between alicyclic and aromatic rings can easily be understood. Introduction of a polar group into a ring system provides a strong preference for the places between the carbon chains and prevents the molecule from going to the inside of the micelle and exerting its "opening" action. With the parent substance the action was the sum of a condensing and opening action, resulting from a distribution equilibrium between the two places available. It is to be expected that this equilibrium will have a different value for various hydrocarbons.

My sincere thanks are due to Prof. H. G. BUNGENBERG DE JONG and Dr H. VELDSTRA. It is doubtful whether the theoretical explanation of the phenomena described would have been developed without the many discussions we had on this subject. From the Research laboratory, Combinatie N.V. en Amsterdamsche, Bandoengsche en Nederlandsche Kinine-fabrieken I received a considerable quantity of cetylsulfate.

Summary.

1. With an oleate as well as a cetylsulfate coacervate it was observed that aromatic ring systems (without polar group) have a greater "condensing" action than comparable alicyclic ones.
2. For compounds with a polar group the situation is exactly reversed.
3. Heterocyclic rings will arrange themselves according to their aromatic character, the activity increasing in the range $N < O < S < C$.
4. In some cases the introduction of a polar group results in a stronger activity than in the parent substance (e.g. naphthol - naphthalene).
5. Starting from the working hypothesis that apolar substances have essentially two places of action within the soap micelles (a. in between the parallel carbon chains and b. between the CH_3 planes) the phenomena described have been explained. A compound with a polar group is held in position a., for substances without a polar group an equilibrium between the amounts present at a. and b. will establish itself. Obviously, this equilibrium will depend on the molecular structure and presumably on the situation in the micelle.

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Mathematics. — *Existence of Stieltjes integrals. I.* By R. F. DENISTON *)
(Ames, Iowa). (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 24, 1949.)

The purpose of this paper is to extend work of others on the left- and right-Cauchy-Stieltjes integrals, the modified integral of Dushnik and the ordinary Stieltjes integral. The present work centers about the proof of the necessary and sufficient conditions for the existence of the integrals in the Pollard-Moore sense. By then employing a theorem of Getchell the corresponding conditions for the existence of integrals in the norm sense are given and compared. Conditions are obtained without the usual restrictions that functions considered have only discontinuities of the first kind.

Interest arises in these integrals because of their relation to the Lebesgue-Stieltjes and other integrals and from the consideration of generalizations of the ordinary Stieltjes integral ¹⁾.

Interval Functions and Sums

We suppose the finite functions $f(x)$ and $g(x)$ defined in the interval $\langle a, b \rangle (= \{x; a \leq x \leq b\})$. The definition of the various kinds of integrals to be considered shall be made to depend on certain functions of an interval, which in turn depend on $f(x)$ and $g(x)$. Thus for $\langle t', t'' \rangle$, an arbitrary subinterval of $\langle a, b \rangle$, we establish a number $U(\langle t', t'' \rangle)$ corresponding to this interval, which we shall call the value of the function of the interval. In the cases to be considered this takes the form:

$$U(\langle t', t'' \rangle) = f(\xi)[g(t'') - g(t')],$$

where ξ is some prescribed point in $\langle t', t'' \rangle$.

The modifications giving rise to different kinds of interval functions now center in the multiplier $f(\xi)$, in particular in the choice of the point ξ . We shall study the following cases:

$$F(\langle t', t'' \rangle) = f(\xi)[g(t'') - g(t')] \quad t \leq \xi \leq t''.$$

This is the sum giving rise to the ordinary Stieltjes integral.

$$F^{(-)}(\langle t', t'' \rangle) = f(t')[g(t'') - g(t')]$$

$$F^{(+)}(\langle t', t'' \rangle) = f(t'')[g(t'') - g(t')]$$

*) The writer is indebted to Dr. J. RIDDER for valuable advice and criticism.

¹⁾ For these considerations see the writer's Master's Thesis, Washington University, St. Louis, Mo., 1948, "Generalizations of the Riemann-Stieltjes Integral".

These give rise to the Left- and Right-Cauchy integrals respectively.

$$F^{(*)}(<t', t''>) = f(\xi) [g(t'') - g(t')] \quad t' < \xi < t''$$

This is the modified sum of Dushnik.

If we wish to leave unspecified the type sum considered in a discussion we use the notation $F^{(\sim)}(<t', t''>)$.

Let now a mode of subdivision which we shall denote by D be formed in the interval $<a, b>$ by the points

$$a = t_0 < t_1 < \dots < t_r = b,$$

and let for an interval function $F^{(\sim)}(<t', t''>)$ the following sum be formed

$$\sum_{i=0}^{r-1} F^{(\sim)}(<t_i, t_{i+1}>) = D \overset{(\sim)}{S} [f, g].$$

We shall be concerned with the limits of collections of such sums.

Types of Limits

The Norm Sense.

A sequence of modes of subdivision $\{D_n\}$ will be called normal if the length of the longest of the subintervals of D_n tends to 0 as n tends to infinity. If for every normal sequence of subdivisions the sequences of sums $\{D_n \overset{(\sim)}{S} [f, g]\}$ converge to a limit common for all normal sequences we shall call the resulting limit the integral in the norm sense and shall denote it by $N^{(\sim)} \int_a^b f dg$.

The Pollard-Moore Sense.

We shall say that $D \supseteq D_0$ when every point of division pertaining to D_0 is a point of division of D , or we shall say that D is a refinement of, or is finer than, D_0 . For a particular manner of definition of $D \overset{(\sim)}{S} [f, g]$ we shall say that the set of such sums has the limit, L , in the Moore-Smith sense if for arbitrary $\epsilon > 0$ it is possible to find a mode of subdivision D_ϵ such that $|D \overset{(\sim)}{S} [f, g] - L| < \epsilon$ for all D such that $D \supseteq D_\epsilon$. The resulting limit will be called the integral in the Pollard-Moore sense and denoted by $\sigma^{(\sim)} \int_a^b f dg$. (The name POLLARD is usually included because the integral studied by him has essentially this definition.)

Summary of Principle Results

In this paper necessary and sufficient conditions are established for the existence of integrals $\int_a^b f dg$ based on several types of interval functions and taken in both the norm and Pollard-Moore senses, and where f is initially restricted only by the condition of being bounded on the interval $<a, b>$ with g a function of bounded variation on this interval. We shall

seek the conditions or restrictions on the behaviour of f with relation to the function g . Because of the amount of work which has been done by others²⁾ much of the present study is concerned with the establishment of the conditions (Theorem 1) for the existence of the integrals taken in the Pollard-Moore or σ -sense for the interval functions $F^{(-)}(< t', t'' >)$ and $F^{(+)}(< t', t'' >)$ which we shall call the Left-Cauchy and the Right-Cauchy integral respectively. These integrals have been studied for the case: f bounded with only simple discontinuities and g non-decreasing in $< a, b >$, by PRICE [7], who shows that if f and g are restricted in this manner both the Left- and Right-Cauchy integrals exist. In the present paper results thought to be new in the form of the necessary and sufficient conditions for the existence of these integrals when f is initially restricted only by the condition of being bounded are obtained.

Using the conditions for the existence of the Cauchy integrals in the σ -sense it is then easy to prove (Theorem 3) the conditions given by SCHAERF [8] for the existence of these integrals in the norm sense. This is accomplished by the employment of a theorem of GETCHELL [3]. Theorem 2 determines a condition of pseudo-additivity for these integrals needed for the application of Getchell's theorem.

Theorem 4 gives the corresponding necessary and sufficient conditions for the Pollard-Moore ordinary integral. These agree with well known results and correspond in accordance with the theorem of Getchell with the result obtained by Bliss for the ordinary Riemann-Stieltjes integral.

Theorem 5 gives necessary and sufficient conditions for the existence of the modified integral of Dushnik, based on the interval function $F^{(*)}(< t', t'' >)$, in the Pollard-Moore sense. Theorem 6 gives the corresponding conditions in the norm sense.

Definitions of Auxiliary Functions.

In the sequel we shall make use of the following definitions:

$(a, b >$ is the half closed interval $\{x; a < x \leq b\}$ similar for $< a, b)$, etc.

h , the saltus function of g (function of the jumps).

$h_l, [h_r]$, the saltus function formed from the left [right] discontinuities of g .

$g_c(x) = g(x) - h(x)$, the continuity function of g .

$g_l(x) = g(x) - h_l(x)$ [$g_r(x) = g(x) - h_r(x)$] the left [right] continuity function of g .

$\overset{(-)}{\Delta} g(x) = g(x) - g(x-0)$.

$\overset{(+)}{\Delta} g(x) = g(x+0) - g(x)$.

$\Delta g(x) = g(x+0) - g(x-0)$. (Similar for $\overset{(-)}{\Delta} h(x)$, $\overset{(+)}{\Delta} h(x)$, etc.)

If I is an arbitrary interval, $< t', t'' >$, $g(I) = g(t'') - g(t')$, and similarly for $h(I)$, $h_l(I)$, etc.

²⁾ See T. H. HILDEBRANDT [4] for bibliography of work prior to 1938. Numbers in brackets refer to Bibliography at end of this paper.

If N is a set of points, $g^*(N)$ will denote the outer g -measure (infinitely additive) of N .

We denote by M a bound for $|f|$ on $\langle a, b \rangle$.

The Left- and Right-Cauchy-Stieltjes Integrals in the Pollard-Moore Sense

We prove in this section the theorem of existence for the Left-Cauchy-Stieltjes integral. A corresponding theorem holds for the Right-Cauchy-Stieltjes integral.

Theorem 1. *In the case of f bounded, g of bounded variation in $\langle a, b \rangle$ for the existence of the integral $\sigma^{(-)} \int_a^b f dg$ the following conditions are necessary and sufficient:*

(a). *In every point of the interval $\langle a, b \rangle$ in which g is discontinuous on the left the function f has a left limit, $f(x-0)$.*

(b). *The set of points in $\langle a, b \rangle$ which are left-sided discontinuities of the function f is a null set with respect to $g_c(x)$ ³⁾, the continuity function of g .*

Proof. Condition (a) is necessary. If not, let t be a point of $\langle a, b \rangle$ such that $\Delta^{(-)} g(t) > 0$ and for which it is supposed that $f(t-0)$ does not exist. It will be shown that there is an ϵ for which there is no D_ϵ in the Moore-Smith sense. For let D_ϵ be such a subdivision for an arbitrary choice of ϵ . Then whenever $D', D'' \supseteq D_\epsilon$ we have

$$|D' \overset{(-)}{S}[f, g] - D'' \overset{(-)}{S}[f, g]| < \epsilon. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let D_ϵ consist of the points

$$a = t_0 < t_1 < \dots < t_r = b.$$

Let the integer m_ϵ be determined by the relation

$$t_{m_\epsilon} < t \leq t_{m_\epsilon+1}.$$

Since $f(t-0)$ does not exist, there will exist two sequences of points $\{x_j\}$ and $\{y_k\}$ such that

$$\lim_{j \rightarrow \infty} x_j = \lim_{k \rightarrow \infty} y_k = t \text{ and } \lim_{j \rightarrow \infty} f(x_j) = f_1, \lim_{k \rightarrow \infty} f(y_k) = f_2,$$

where $f_1 - f_2 = 2p > 0$. Integers j^* and k^* can now be found sufficiently large so that the following several relations are satisfied:

$$f(x_{j^*}) - f(y_{k^*}) > p, \quad t_{m_\epsilon} < x_{j^*} < y_{k^*} < t,$$

$|g(x_{j^*}) - g(t-0)| < \eta/2$, $|g(y_{k^*}) - g(t-0)| < \eta/2$ and $|g(x_{j^*}) - g(y_{k^*})| < \eta/2$ for arbitrary $\eta > 0$.

³⁾ By definition a null set with respect to $g_c(x)$ is a set whose measure with respect to the total variation of $g_c(x)$ is equal to 0.

Now let D'_ϵ be composed of the points of D_ϵ and the points t and x_{j*} ; D''_ϵ of the points of D_ϵ and the points t and y_{k*} . We have

$$\begin{aligned} D' \overset{(-)}{S}[f, g] - D'' \overset{(-)}{S}[f, g] &= f(t_{m_\epsilon}) [g(x_{j*}) - g(t_{m_\epsilon})] + f(x_{j*}) [g(t) - g(x_{j*})] \\ &\quad - f(t_{m_\epsilon}) [g(y_{k*}) - g(t_{m_\epsilon})] - f(y_{k*}) [g(t) - g(y_{k*})] \\ &= [f(x_{j*}) - f(y_{k*})] \cdot [g(t) - g(y_{k*})] \\ &\quad + [f(t_{m_\epsilon}) - g(x_{j*})] \cdot [g(x_{j*}) - g(y_{k*})]. \end{aligned}$$

Therefore, by the above choice of j^* and k^* and if M is a bound for $|f|$ on $\langle a, b \rangle$

$$|D' \overset{(-)}{S} - D'' \overset{(-)}{S}| \cong p(\overset{(-)}{\Delta} g(t) - \eta) - M\eta.$$

The left-hand side of this inequality is clearly $> \frac{p}{2} \overset{(-)}{\Delta} g(t)$ for a suitably small choice of η . Now if ϵ has been chosen less than $\frac{p}{2} \overset{(-)}{\Delta} g(t)$ we have a contradiction with (1). Hence condition (a) is necessary.

Condition (b) is necessary. Since for all integrals to be considered we have the property $\int f d(g_1 + g_2) = \int f dg_1 + \int f dg_2$ and since as is well known we can express g_ϵ as the sum of functions g_1 and g_2 , where g_1 is monotonic non-decreasing, and g_2 monotonic non-increasing, no loss in generality will result if we suppose g_ϵ non-decreasing on $\langle a, b \rangle$.⁴⁾ Now we suppose on the contrary that the integral in question exists while also the set L of the points in $\langle a, b \rangle$ which are points of left-side discontinuity has positive outer g_ϵ -measure. We denote by L_p the set of points, x , for which there is in every neighborhood a point y lying to the left of x for which $|f(x) - f(y)| > p$. Since $L = \sum_{n=1}^{\infty} L_{1/n}$, and since $0 < g_\epsilon^* \{L\} \cong \sum_{n=1}^{\infty} g_\epsilon^* \{L_{1/n}\}$, there is a number $p > 0$ for which $g_\epsilon^* \{L_p\} = m_p > 0$.

Since it is assumed that the integral $\sigma^{(-)} \int_a^b f dg$ exists, let for arbitrary $\eta > 0$, D_η designate a mode of subdivision having the property that for all D', D'' for which $D', D'' \supseteq D_\eta$ it is true that $|D' \overset{(-)}{S}[f, g] - D'' \overset{(-)}{S}[f, g]| < \eta$. Suppose that D_η is given by $\{a = t_0 < t_1 < \dots < t_{r_\eta} = b\}$ and consider the set $\sum_{i=1}^{r_\eta-1} \langle t_i - \delta, t_i + \delta \rangle$. By the uniform continuity of g_ϵ on $\langle a, b \rangle$ we are assured of a δ such that if x and y satisfy $|x - y| < 2\delta$, we have $|g_\epsilon(x) - g_\epsilon(y)| < \epsilon/2r_\eta$. Using this δ in the definition of the set $L'_p = L_p - \sum_{i=1}^{r_\eta-1} \langle t_i - \delta, t_i + \delta \rangle$ we have that the outer g_ϵ -measure of the set is greater than $m_p - \epsilon/2$. Hence by the meaning of outer measure we

⁴⁾ Cf. SCHAERF [8], p. 84, 90—92.

can cover the set L'_p by a finite number n_1 of non-overlapping intervals $I_i^{(1)}$, $i = 1, 2, \dots, n_1$, satisfying

$$\sum_{i=1}^{n_1} g_c(I_i^{(1)}) > \sum_{i=1}^{n_1} g_c^*(I_i^{(1)} \cdot L'_p) > m_p - \epsilon$$

and having no points in common with the set $\sum_{i=1}^{r_{\eta}-1} \langle t_i - \partial, t_i + \partial \rangle$.

Let ∂_1 be the ∂ of uniform continuity for ϵ/n_1 and let us move each of the left end points of the above intervals this distance to the right. If any interval, $I_i^{(1)}$, is shorter than ∂_1 or, after the changing of its left end point, contains no points of L'_p , we omit it. We thus form a set of non-overlapping intervals $I_i^{(2)}$ at distance $\geq \partial_1$ apart and for which

$$\sum_{i=1}^{n_2} g_c(I_i^{(2)}) > \sum_{i=1}^{n_2} g_c^*(I_i^{(2)} \cdot L'_p) > m_p - 2\epsilon.$$

Let a_i be the new left end points. We now have the following possibilities:

(i). To the right of a_i there is in the interior of $I_i^{(2)}$ a first point a'_i of L_p , or

(ii). In a_i or to the right of a_i in the interior of $I_i^{(2)}$ there is a first point a_i^* of those having the property of being a limit point on the right of points of L_p .

If (i) holds we take a'_i as x_i . If (ii) holds there is a point a_i'' of L_p such that $a_i'' - a_i^* < \partial_1$ and hence $g_c(a_i'') - g_c(a_i^*) < \epsilon/n_2$. Designate this point as x_i . In each case let the right end point be designated as y_i . Let $\langle x_i, y_i \rangle = I_i^{(3)}$, $i = 1, 2, \dots, n_3 = n_2$. We have

$$\sum_{i=1}^{n_3} g_c(I_i^{(3)}) > \sum_{i=1}^{n_3} g_c^*(I_i^{(3)} \cdot L_p) > m_p - 3\epsilon.$$

The expression on the right is greater than $m_p/2 = m > 0$ if ϵ has been chosen less than $m_p/6$.

In resume the last set of intervals satisfy the following conditions: They are at positive distances from each other and from any point of D_η . The left end points, x_i , are points of L_p . Then there is a $\partial > 0$ and either a point x'_i or a point x_i'' such that

$$(iii) \quad 0 < x_i - x'_i < \partial \quad \text{and} \quad f(x_i) - f(x'_i) > p \quad \text{or}$$

$$(iv) \quad 0 < x_i - x_i'' < \partial \quad \text{and} \quad f(x_i'') - f(x_i) > p.$$

Let the modes of subdivision D' and D'' be formed as follows: Both D' and D'' have all the points of D_η and all $\{y_i\}$.

If (iii) holds for i let x_i be in D' and not in D'' ,

x'_i be in D'' and not in D' .

If (iv) holds for i let x_i be in D'' and not in D' ,

x_i'' be in D' and not in D'' .

Then a calculation gives

$$D' \overset{(-)}{S} [f, g] - D'' \overset{(-)}{S} [f, g] \geq p \cdot m,$$

which is greater than η if η has been chosen to be $< p \cdot m$. Since both D' and D'' are refinements of D_η this result contradicts the definition of D_η .

The conditions are sufficient.

For integrals in both senses the equality

$${}^{(-)}\int_a^b f dh_r + {}^{(-)}\int_a^b f dh_l + {}^{(-)}\int_a^b f dg_c = {}^{(-)}\int_a^b f dg$$

follows from the same property for the corresponding sums. From the existence of the three integrals on the left it will follow that the integral on the right exists. We now show the sufficiency of the given conditions for the existence of $\sigma^{(-)}\int_a^b f dh_l$, $\sigma^{(-)}\int_a^b f dh_r$ and $\sigma^{(-)}\int_a^b f dg_c$. We shall assume h_l and h_r non-decreasing as this assumption results in no loss of generality.

(A). *The case of $\sigma^{(-)}\int_a^b f dh_l$.*

Let the points $\{x_i\}$ at which h_l is discontinuous be supposed to be numbered in the order of decreasing magnitude of

$$\Delta^{(-)} h_l(x_i) = h_l(x_i) - h_l(x_i - 0).$$

Let for any m , $h_l^{(m)}$ be the saltus function for which

$$\Delta^{(-)} h_l^{(m)}(x_i) = \Delta^{(-)} h_l(x_i) \quad (i = 1, 2, \dots, m)$$

and for which $\Delta^{(-)} h_l^{(m)} = 0$ for all other x . Let us choose an m so that

$$2M \sum_{i=m+1}^{\infty} \Delta^{(-)} h_l(x_i) < \epsilon/4.$$

By the assumption of the existence of a left limit, $f(x_i - 0)$ at x_i , $i = 1, 2, \dots, m$, we can find points $\{y_i\}$ such that y_i is between x_i and the nearest point of the set $\{x_i\}$ lying to the left of x_i and such that

$$|f(x_i - 0) - f(y_i)| < \frac{\epsilon}{4 \sum_{i=1}^m \Delta^{(-)} h_l^{(m)}(x_i)}$$

and such that a similar inequality holds for $y_i < x \leq x_i$. Let D_ϵ consist of the $\{x_i\}$ and $\{y_i\}$ together with the points a and b . It is clear that the intervals $\{< y_i, x_i >\}$ are the only one contributing to $D_\epsilon \overset{(-)}{S} [f, h_l^{(m)}]$ and that by the choice of the y_i we have

$$|D_\epsilon \overset{(-)}{S} [f, h_l^{(m)}] - \sum_{i=1}^m f(x_i - 0) \Delta^{(-)} h_l^{(m)}(x_i)| < \epsilon/4. \quad . \quad . \quad . \quad (1)$$

Also by the choice of the y_i , for any two refinements D' , D'' of D_ϵ we have

$$|D' \overset{(-)}{S} [f, h_l^{(m)}] - D_\epsilon \overset{(-)}{S} [f, h_l^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad . \quad (2)$$

$$|D'' \overset{(-)}{S} [f, h_l^{(m)}] - D_\epsilon \overset{(-)}{S} [f, h_l^{(m)}]| < \epsilon/4. \quad . \quad . \quad . \quad . \quad (3)$$

By the choice of m we have

$$|D' \overset{(-)}{S}[f, h_l] - D' \overset{(-)}{S}[f, h_l^{(m)}]| < \epsilon/4, \dots \quad (4)$$

$$|D'' \overset{(-)}{S}[f, h_l] - D'' \overset{(-)}{S}[f, h_l^{(m)}]| < \epsilon/4, \dots \quad (5)$$

and also

$$\left| \sum_{i=1}^m f(x_i - 0) \overset{(-)}{\Delta} h_l(x_i) - \sum_{i=1}^{\infty} f(x_i - 0) \overset{(-)}{\Delta} h_l(x_i) \right| < \epsilon/4, \dots \quad (6)$$

From (2), (3), (4), and (5) we have

$$|D' \overset{(-)}{S}[f, h_l] - D'' \overset{(-)}{S}[f, h_l]| < \epsilon,$$

which establishes that D_ϵ is a mode of subdivision for ϵ in accordance with the Pollard-Moore meaning of limits.

From (1), (2), (4), and (6) we have the additional result

$$\sigma^{(-)} \int_a^b f dh_l = \sum_{i=1}^{\infty} f(x_i - 0) \overset{(-)}{\Delta} h_l(x_i),$$

where x_i runs over the points of left discontinuity of g (or h_l).

(B). The case of $\sigma^{(-)} \int_a^b f dh_r$.

In exact analogy to $h_l^{(m)}$ we define $h_r^{(m)}$ and choose m so that

$$\left. \begin{aligned} 2M \sum_{i=m+1}^{\infty} \overset{(+)}{\Delta}(x_i) &< \epsilon/2, \text{ that is} \\ \left| \sum_{i=1}^{\infty} f(x_i) \overset{(+)}{\Delta} g_r(x_i) - \sum_{i=1}^m f(x_i) \overset{(+)}{\Delta} g_r(x_i) \right| &< \epsilon/2. \end{aligned} \right\} \dots \quad (8)$$

In the present case let D_ϵ consist of $\{x_i\}$ $i = 1, 2, \dots, m$, and a and b . Then we have for D_ϵ

$$D_\epsilon \overset{(-)}{S}[f, h_r^{(m)}] = \sum_{i=1}^m f(x_i) \overset{(+)}{\Delta} h_r(x_i).$$

It is easy to see that for $D', D'' \supseteq D_\epsilon$

$$D' \overset{(-)}{S}[f, h_r^{(m)}] = D'' \overset{(-)}{S}[f, h_r^{(m)}] = D_\epsilon \overset{(-)}{S}[f, h_r^{(m)}].$$

By the choice of m

$$|D' \overset{(-)}{S}[f, h_r] - D' \overset{(-)}{S}[f, h_r^{(m)}]| < \epsilon/2$$

or

$$|D' \overset{(-)}{S}[f, h_r] - \sum_{i=1}^m f(x_i) \overset{(+)}{\Delta} h_r(x_i)| < \epsilon/2, \dots \quad (9)$$

(8) and (9) yield the result

$$|D' \overset{(-)}{S}[f, h_r] - \sum_{i=1}^{\infty} f(x_i) \overset{(+)}{\Delta} h_r(x_i)| < \epsilon$$

showing that $\sigma^{(-)} \int_a^b f dh_r$ exists and

$$\sigma^{(-)} \int_a^b f dh_r = \sum_{i=1}^{\infty} f(x_i) \Delta^{(+)} h_r(x_i)$$

where x_i runs over the points of right discontinuity of h_r (or of g).

(C). *The case of $\sigma^{(-)} \int_a^b f dg_c$.*

We form a sequence $\{f_n\}$ of step functions defined as follows: For

$$a + (b-a)i/n \leq x < a + (b-a)(i+1)/n, \quad i = 0, 1, \dots, n-1,$$

let $f_n(x) = f(a + (b-a)i/n)$, $n = 1, 2, \dots$

The functions $f_n(x)$ converge almost everywhere (g_c) in $\langle a, b \rangle$, f being left continuous at all these points except a null set. Moreover, $|f_n(x)| < M$ for x in $\langle a, b \rangle$. Then, with reference to Mc Shane's "Integration", [5], page 276, Theorem 52. 20, the Lebesgue-Stieltjes integral $\int_{\langle a, b \rangle} f dg$ exists, and we have

$$\lim_{n \rightarrow \infty} \int_{\langle a, b \rangle} f_n dg_c = \int_{\langle a, b \rangle} f dg_c.$$

Hence for any ϵ we can find an n' such that

$$|\int_{\langle a, b \rangle} f_{n'} dg_c - \int_{\langle a, b \rangle} f dg_c| < \epsilon.$$

Let us take for the mode, D_ϵ ,

$$a = t_0^{(n')} < t_1^{(n')} \dots < t_{n'}^{(n')} = b,$$

where

$$t_i^{(n')} = a + (b-a)i/n', \quad i = 0, 1, 2, \dots, n',$$

and for this D_ϵ , we have

$$\begin{aligned} D_\epsilon S^{(-)}[f, g_c] &= \sum_{i=1}^{n-1} f(t_i) [g_c(t_{i+1}) - g_c(t_i)] \\ &= \sum_{i=1}^{n-1} f_{n'}(x) g_c^*(\langle t_i, t_{i+1} \rangle) = \int_{\langle a, b \rangle} f_{n'} dg_c. \end{aligned}$$

Hence for D_ϵ

$$|D_\epsilon S^{(-)}[f, g_c] - \int_{\langle a, b \rangle} f dg_c| < \epsilon$$

showing that a limit is reached in the Pollard-Moore manner, and we have

$$\sigma^{(-)} \int_a^b f dg_c = \int_{\langle a, b \rangle} f dg_c.$$

Mathematics. — *Existence of Stieltjes integrals. II.* By R. F. DENISTON (Ames, Iowa). (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of September 24, 1949.)

The Left- and Right-Cauchy-Stieltjes Integral in the Norm Sense

We shall employ the definition of pseudo-additivity of an interval function and the theorem given by GETCHELL [3] which gives necessary and sufficient conditions for the existence of an integral in the norm sense.

For a specified interval function $\overset{(\sim)}{F}(I)$ we say that $\overset{(\sim)}{F}(I)$ is pseudo-additive at a point z if for $x < z < y$

$$\lim_{x \rightarrow z, y \rightarrow z} \text{Lub} |\overset{(\sim)}{F}(\langle x, y \rangle) - \overset{(\sim)}{F}(\langle x, z \rangle) - \overset{(\sim)}{F}(\langle z, y \rangle)| = 0.$$

An interval function will be said to be pseudo-additive on an interval $\langle a, b \rangle$ if it is pseudo-additive at each point of $\langle a, b \rangle$.

Theorem of Getchell. For a specified interval function, $\overset{(\sim)}{F}(I)$, for the existence of the integral $N^{(\sim)} \int_a^b f dg$ the following conditions are necessary and sufficient:

$$\sigma^{(\sim)} \int_a^b f dg \text{ exist; and } \overset{(\sim)}{F}(I) \text{ be pseudo-additive on } \langle a, b \rangle.$$

Theorem 2. *The condition of pseudo-additivity for $\overset{(-)}{F}(I)$ is equivalent to the following:*

Condition (q): In each point of (a, b) in which the function g is discontinuous on the right the function f is continuous on the left.

Proof. A calculation gives

$$\begin{aligned} & |\overset{(-)}{F}(\langle x, y \rangle) - \overset{(-)}{F}(\langle x, z \rangle) - \overset{(-)}{F}(\langle z, y \rangle)| = \\ & = |f(x)[g(y) - g(x)] - f(x)[g(z) - g(x)] - f(z)[g(y) - g(z)]| \\ & = |g(y) - g(z)| \cdot |f(x) - f(z)|. \end{aligned}$$

The condition of pseudo-additivity is hence equivalent to:

Condition (p):

$$\lim_{\substack{x \rightarrow z, x < z \\ y \rightarrow z, y > z}} \text{Lub} |g(y) - g(z)| \cdot |f(z) - f(x)| = 0.$$

We now show that *condition (p) implies condition (q)*, by showing that if (q) is not satisfied (p) is not. Let z be a point at which $|g(z+0) - g(z)| > 0$, and at which f is not left-continuous. Then there is a sequence

of points $\{y_n\}$, $y_n \rightarrow z$, $y_n > z$, $n=1, 2, \dots$ such that $|g(y_n) - g(z)| > \Delta > 0$, and a sequence of points $\{x_n\}$, $x_n \rightarrow z$, $x_n < z$, $n=1, 2, \dots$ such that $|f(z) - f(x_n)| > m$. Hence

$$\lim_{\substack{x \rightarrow z, x < z \\ y \rightarrow z, y > z}} \text{Lub } |g(y) - g(z)| \cdot |f(z) - f(x)| > 0, \text{ and condition (p) is}$$

not realized.

We show that condition (q) implies condition (p). For a point of $(a, b >$ at which g is continuous on the right

$$\lim_{y \rightarrow z, y > z} (g(y) - g(z)) = 0;$$

hence condition (p) is satisfied a priori. For a point of $(a, b >$ at which g is discontinuous on the right condition (q) requires

$$\lim_{x \rightarrow z, x < z} (f(x) - f(z)) = 0,$$

which gives immediately condition (p).

Theorem 3. In the case of f bounded, g of bounded variation in $< a, b >$ for the existence of $N^{(-)} \int_a^b f dg$ the following conditions are necessary and sufficient:

(a'). Same as (a) of Theorem 1.

(b'). The set of points in $(a, b >$ which are left-sided discontinuities of the function f is a null set with respect to the left-side continuity function, g_l , of g .

(This theorem has been proved by SCHAERF [8].)

Proof. In accordance with the theorem of Getchell it is sufficient to show that condition (b') is equivalent to the totality of condition (b) of theorem 1 and condition (q) of theorem 2. We consider separately several sets of points which together exhaust $(a, b >$.

For points of $(a, b >$ in which f is continuous on the left conditions (b) and (b') make no assertion, and condition (q) is trivially satisfied.

Let the set L_1 consist of points of $(a, b >$ in which f is discontinuous on the left and g is discontinuous on the right. If both (b) and (q) hold for L_1 , by (q) f is left-continuous whenever g is right-discontinuous; and hence L_1 is null. This implies (b'). On the other hand if (b') holds L_1 is a null set with respect to g_l . This requires that L_1 have no points in which g is right-discontinuous. Then this also requires L_1 is null, which gives both (b) and (q) true for the set L_1 .

Let the set L_2 consist of points of $(a, b >$ in which f is discontinuous on the left and g is continuous on the right. If z is a point of L_2

$$\Delta g_c(z) = \Delta g_l(z).$$

Hence the g_c -measure and g_l -measure of the point is the same. Then (b) is the same as (b') for L_2 . Condition (q) does not concern L_2 .

The sets considered exhaust the points of $(a, b >$, and thus the proof is complete.

The Stieltjes Integral in the Pollard-Moore Sense

Theorem 4. *In the case of f bounded, g of bounded variation in $< a, b >$ for the existence of the integral $\sigma \int_a^b f dg$ the following conditions are necessary and sufficient:*

(a''). *On each side of each point of (a, b) and the right side of a and left side of b if g is discontinuous the function f is continuous.*

(b''). *The points of (a, b) in which f is discontinuous is a null set with respect to the continuity function of g .*

Proof. *The conditions are necessary.*

Firstly, it is necessary that both $\sigma^{(-)} \int_a^b f dg$ and $\sigma^{(+)} \int_a^b f dg$ exist. If we let L and R be respectively the set of left and right-sided discontinuities of f and N be the set of discontinuities of f , it is required by (b) of Theorem 1 that L is a null set with respect to g_c in order that $\sigma^{(-)} \int_a^b f dg$ exist. A corresponding condition for the existence of $\sigma^{(+)} \int_a^b f dg$ gives that R is a null set with respect to g_c . Hence N is a null set with respect to g_c , and this is condition (b'').

From (a) of theorem 1 and the corresponding condition for the existence of $\sigma^{(+)} \int_a^b f dg$ it follows immediately that at each "side" mentioned in (a'') f has a sidewise limit. If f is not sidewise continuous suppose that at a point, z , of (e.g.) left discontinuity of g (and hence of h_l) that $f(z-0)$ exists and is different from $f(z)$. Let D_ϵ be a subdivision for ϵ in the sense that whenever $D', D'' \supseteq D_\epsilon$

$$|D' S[f, h_l] - D'' S[f, h_l]| < \epsilon.$$

Such a D_ϵ is guaranteed to exist by the existence of $\sigma \int_a^b f dh_l$. Now let D' have all the points of D_ϵ , the point z , and as its first point to the left of z a point x for which $|f(x) - f(z-0)| < \frac{1}{2} |f(z) - f(z-0)|$, and $|h_l(x) - h_l(z-0)| < \frac{1}{2} |h_l(z) - h_l(z-0)|$. Let the sum $D' S_0[f, h_l]$ contain the same terms as $D' S^{(-)}[f, h_l]$ with the exception of the term $f(x) \cdot [h_l(z) - h_l(x)]$ which may be replaced by $f(z) \cdot [h_l(z) - h_l(x)]$. A calculation gives

$$\begin{aligned} |D' S^{(-)}[f, h_l] - D' S_0[f, h_l]| &= |f(z) - f(x)| \cdot |h_l(z) - h_l(x)| \\ &= |f(z) - f(z-0) - (f(x) - f(z-0))| \cdot \\ &\quad |h_l(z) - h_l(z-0) - (h_l(x) - h_l(z-0))| \\ &\cong \frac{1}{4} |f(z) - f(z-0)| \cdot |h_l(z) - h_l(z-0)|. \end{aligned}$$

This difference is greater than ϵ for a suitable choice of ϵ . This result contradicts the definition of D_ϵ for such an ϵ . Similar results are obtained if we suppose a point z of right discontinuity of g such that

$$f(z) \neq f(z+0).$$

The conditions are sufficient.

As in the proof of Theorem 1 we make use of the equation

$$\sigma \int_a^b f dh_l + \sigma \int_a^b f dh_r + \sigma \int_a^b f dg_c = \sigma \int_a^b f dg,$$

and show that each integral on the left exists. We again assume without loss of generality that h_l and h_r are non-decreasing.

(A). *The case of $\sigma \int_a^b f dh_l$.*

Defining $h_l^{(m)}$ as in the proof of Theorem 1 we choose an m so that

$$2M \sum_{i=m+1}^{\infty} \bar{\Delta} h_l(x_i) < \epsilon/4.$$

We can find points $\{y_i\}$ ($i = 1, 2, \dots, m$) such that y_i is between x_i and the nearest point of the set $\{x_i\}$ lying to the left of x_i and such that by the left-continuity of f at x_i

$$|f(\xi_i) - f(x_i)| < \frac{\epsilon}{4 \sum_{i=1}^m \bar{\Delta} h_l^{(m)}(x_i)}$$

whenever $y_i \leq \xi_i \leq x_i$. Let D_ϵ consist of the $\{x_i\}$ and $\{y_i\}$ together with the points a and b . Only the intervals $\{< y_i, x_i >\}$ contribute to $D_\epsilon S[f, h_l^{(m)}]$, and by the choice of y_i we have for $y_i \leq \xi_i \leq x_i$

$$|D_\epsilon S[f, h_l^{(m)}] - \sum_{i=1}^m f(x_i) \bar{\Delta} h_l^{(m)}(x_i)| < \epsilon/4. \quad . \quad . \quad . \quad (1)$$

Also by the choice of the y_i , for any two refinements D', D'' of D_ϵ we have

$$|D' S[f, h_l^{(m)}] - D_\epsilon S[f, h_l^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad . \quad (2)$$

$$|D'' S[f, h_l^{(m)}] - D_\epsilon S[f, h_l^{(m)}]| < \epsilon/4. \quad . \quad . \quad . \quad . \quad (3)$$

By the choice of m we have

$$|D' S[f, h_l] - D' S[f, h_l^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad . \quad (4)$$

$$|D'' S[f, h_l] - D'' S[f, h_l^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad . \quad (5)$$

and also

$$\left| \sum_{i=1}^m f(x_i) \bar{\Delta} h_l(x_i) - \sum_{i=1}^{\infty} f(x_i) \bar{\Delta} h_l(x_i) \right| < \epsilon/4. \quad . \quad . \quad . \quad (6)$$

From (2), (3), (4), and (5) we have

$$|D' S[f, h_l] - D'' S[f, h_l]| < \epsilon$$

which establishes that D_ϵ is a mode of subdivision for ϵ in accordance with the Pollard-Moore limit theory. From (1), (2), (4), and (6) we have the additional result

$$\sigma \int_a^b f dh_l = \sum_{i=1}^{\infty} f(x_i) \bar{\Delta} h_l(x_i)$$

where x_i runs over the points of left discontinuity of h_l (or of g).

(B). The case of $\sigma \int_a^b f dh_r$.

Defining $h_r^{(m)}$ as in the proof of theorem 1 we choose m so that

$$2M \sum_{i=m+1}^{\infty} \Delta^{(+)} h_r(x_i) < \epsilon/4.$$

In this case we find points $\{y_i\}$, $i = 1, 2, \dots, m$, such that y_i is between x_i and the nearest point of the set $\{x_i\}$ lying to the right of x_i and such that by the right-continuity of f at x_i

$$|f(\xi_i) - f(x_i)| < \frac{\epsilon}{4 \sum_{i=1}^m \Delta^{(+)} h_r^{(m)}(x_i)}$$

whenever $x_i \leq \xi_i \leq y_i$. Letting D_ϵ consist of the $\{x_i\}$ and $\{y_i\}$ together with the points a and b the proof follows almost exactly like (a) above. We have also the additional result:

$$\sigma \int_a^b f dh_r = \sum_{i=1}^{\infty} f(x_i) \Delta^{(+)} h_r(x_i)$$

where x_i runs over the points of right discontinuity of h_r (or of g).

(C). The case of $\sigma \int_a^b f dg_c$.

The conditions are sufficient by a theorem of Bliss (see below) for the existence of the integral in the norm sense or $N \int_a^b f dg_c$. This guarantees existence in the σ -sense.

Remark on the Ordinary Stieltjes Integral.

For the integral $N \int_a^b f dg$ based on $F(< t', t'' >) = f(\xi)[g(t'') - g(t')]$ ($t' \leq \xi \leq t''$) GETCHELL gives as the condition of pseudo-additivity the following:

$f(x)$ and $g(x)$ have no common point of discontinuity.

This condition and theorem 4 are seen to be in agreement (in the sense of Getchell's theorem) with the following condition given by BLISS [1] for the ordinary integral:

A necessary and sufficient condition that the norm or Riemann-Stieltjes integral $N \int_a^b f dg$, g of bounded variation, exist is that the total variation of g on the set of the discontinuities of f be zero.

The Modified Integral of Dushnik in the Pollard-Moore Sense

Theorem 5. In the case of f founded, g of bounded variation in $< a, b >$ for the existence of the integral $\sigma^{(*)} \int_a^b f dg$ (the Dushnik integral in the Pollard-Moore sense) the following conditions are necessary and sufficient:

(a*). On each side of each point of (a, b) and the right side of a and left side of b if g is discontinuous the function f has a sidewise limit.

(b*). Same as (b''). (i.e. the points of (a, b) in which f is discontinuous is a null set with respect to the continuity function of g .)

Proof. *The conditions are necessary.*

(a*) was given by HILDEBRANDT [4] to be necessary and follows from the necessity of sidewise pseudo-additivity of $F^{(*)}$ as given by GETCHELL.

Condition (b) is necessary.*

We suppose without loss of generality that g_c is non-decreasing in $\langle a, b \rangle$. In order to establish a contradiction we suppose that the set N of discontinuities of f in (a, b) has positive outer g_c -measure, but that the integral $\sigma^{(*)} \int_a^b f dg$ exists. Then for arbitrary $\eta > 0$ there is a mode D_η having the property that for all D', D'' satisfying $D', D'' \supseteq D_\eta$ it is true that

$$|D' \overset{(*)}{S}[f, g] - D'' \overset{(*)}{S}[f, g]| < \eta.$$

We denote by N_p the set of points, x , for which there is in every neighborhood a point y such that $|f(x) - f(y)| > p$. Since $N = \sum_{n=1}^{\infty} N_{1/n}$ and since $0 < g_c^* \{N\} \leq \sum_{n=1}^{\infty} g_c^* \{N_{1/n}\}$ there is a number $p > 0$ for which $g_c^* (N_{1/p}) = m_p > 0$.

Let D_η above be given by

$$a = t_0 < t_1 < \dots < t_{r_\eta} = b \text{ and consider the set } \sum_{i=1}^{r_\eta-1} \langle t_i - \delta, t_i + \delta \rangle.$$

By the uniform continuity of g_c on $\langle a, b \rangle$ we are assured of a δ such that if x and y satisfy $|x - y| < 2\delta$ we have $|g_c(x) - g_c(y)| < \frac{\epsilon}{2r_\eta}$. Using this δ we have that the outer g_c -measure of N'_p , by which we denote $N_p - \sum_{i=1}^{r_\eta-1} \langle t_i - \delta, t_i + \delta \rangle$, is greater than $m_p - \epsilon/2$. Hence by the meaning of outer measure we can cover the set N'_p by a finite number n_1 of non-overlapping intervals $I_i^{(1)} \equiv \langle x_i, y_i \rangle$, $i = 1, 2, \dots, n_1$, satisfying

$$\sum_{i=1}^{n_1} g_c(I_i^{(1)}) > \sum_{i=1}^{n_1} g_c^*(I_i^{(1)} \cdot N'_p) > m_p - \epsilon,$$

and having no points in common with the set $\sum_{i=1}^{r_\eta-1} \langle t_i - \delta, t_i + \delta \rangle$.

Let δ_1 be the δ of uniform continuity for $\epsilon/2n_1$ and let us take points x'_i at a distance δ_1 to the right of x_i and also points y'_i at a distance δ_1 to the left of y_i . If any interval had length less than $2\delta_1$ or if $\langle x'_i, y'_i \rangle$ contains no points of N_p we omit it. Let $\{I_i^{(2)}\}$, $i = 1, 2, \dots, n_2$, be the new (renumbered) set of intervals $\langle x'_i, y'_i \rangle$. We have

$$\sum_{i=1}^{n_2} g_c(I_i^{(2)}) > \sum_{i=1}^{n_2} g_c^*(I_i^{(2)} \cdot N'_p) > m_p - 2\epsilon.$$

The expression on the right is greater than $m_p/2 = m > 0$ if ϵ has been chosen less than $m_p/4$. We now have the following possibilities:

- (i). To the right of x'_i there is in $\langle x'_i, y'_i \rangle$ a first point t_i of N_p , or
 (ii). In x'_i or to the right of x'_i in $\langle x'_i, y'_i \rangle$ there is a first point of those having the property of being a limit point on the right of points of N_p .

If (i) holds take t_i as z_i . If (ii) holds there is in (x'_i, y'_i) a point t_i of N_p . Take this t_i as z_i .

By the nature of the points of N_p , there is near z_i in each interval (x'_i, y'_i) either a point z'_i or a point z''_i satisfying

(iii). $f(z'_i) - f(z_i) > p$, or

(iv). $f(z_i) - f(z'_i) > p$.

Let the modes of subdivision D' and D'' be formed as follows: Both D' and D'' have all points of D_ϵ and the points $\{x_i\}$ and $\{y_i\}$ and differ only in the point taken for ξ_i in forming the sums

$$F^{(*)} = \Sigma f(\xi_i) [g(y_i) - g(x_i)].$$

If (iii) holds take z'_i as ξ_i for D' and z_i as ξ_i for D'' . If (iv) holds take z_i as ξ_i for D' and z'_i as ξ_i for D'' . Then a calculation gives:

$$D' \overset{(*)}{S} [f, g] - D'' \overset{(*)}{S} [f, g] \geq p \cdot m,$$

which is greater than η if η has been chosen to be $< p \cdot m$. Since both D' and D'' are refinements of D_η , this result contradicts the definition of D_η .

The conditions are sufficient.

As in the proof of Theorem 1 we make use of the equation

$$\sigma^{(*)} \int_a^b f dh_r + \sigma^{(*)} \int_a^b f dh_l + \sigma^{(*)} \int_a^b f dg_c = \sigma^{(*)} \int_a^b f dg$$

and show that each integral on the left exists. We again assume without loss of generality that h_l and h_r are non-decreasing.

(A). *The case of $\sigma^{(*)} \int_a^b f dh_l$.*

Defining $h_l^{(m)}$ as in the proof of theorem 1 we choose an m so that

$$2M \sum_{i=m+1}^{\infty} \overset{(-)}{\Delta} h_l(x_i) < \epsilon/4.$$

We can find points $\{y_i\}$, $i = 1, 2, \dots, m$, such that y_i is between x_i and the nearest point of the set $\{x_i\}$ lying to the left of x_i and such that by the existence of $f(x_l - 0)$

$$|f(\xi_i) - f(x_l - 0)| < \frac{\epsilon}{4 \sum_{i=1}^m \overset{(-)}{\Delta} h_l^{(m)}(x_i)}$$

whenever $y_i \leq \xi_i < x_i$. Let D_ϵ consist of the $\{x_i\}$ and $\{y_i\}$ together with the points a and b . Only the intervals $\{\langle y_i, x_i \rangle\}$ contribute to $D_\epsilon \overset{(*)}{S} [f, h_l^{(m)}]$ and by the choice of y_i we have for $y_i \leq \xi_i \leq x_i$

$$|D_\epsilon \overset{(*)}{S} [f, h_l^{(m)}] - \sum_{i=1}^m f(x_i - 0) \overset{(-)}{\Delta} h_l^{(m)}(x_i)| < \epsilon/4. \quad . \quad . \quad . \quad (1)$$

Also by the choice of the y_i , for any two refinements D', D'' of D_ϵ we have

$$|D' \overset{(*)}{S}[f, h_i^{(m)}] - D_\epsilon \overset{(*)}{S}[f, h_i^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad (2)$$

$$|D'' \overset{(*)}{S}[f, h_i^{(m)}] - D_\epsilon \overset{(*)}{S}[f, h_i^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad (3)$$

By the choice of m we have

$$|D' \overset{(*)}{S}[f, h_i] - D' \overset{(*)}{S}[f, h_i^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad (4)$$

$$|D'' \overset{(*)}{S}[f, h_i] - D'' \overset{(*)}{S}[f, h_i^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad (5)$$

and also

$$|\sum_{i=1}^m f(x_i - 0) \overset{(-)}{\Delta} h_i(x_i) - \sum_{i=1}^\infty f(x_i - 0) \overset{(-)}{\Delta} h_i(x_i)| < \epsilon/4, \quad . \quad . \quad (6)$$

From (2), (3), (4), and (5) we have

$$|D' \overset{(*)}{S}[f, h_i] - D'' \overset{(*)}{S}[f, h_i]| < \epsilon$$

which establishes that D_ϵ is a mode of subdivision for ϵ in accordance with the Pollard-Moore limit theory. From (1), (2), (4), and (6) we have the additional result

$$\sigma^{(*)} \int_a^b f dh_i = \sum_{i=1}^\infty f(x_i - 0) \overset{(-)}{\Delta} h_i(x_i)$$

where x_i runs over the points of left discontinuity of h_i (or of g).

(B). *The case of $\sigma^{(*)} \int_a^b f dh_r$.*

Defining $h_r^{(m)}$ as in the proof of theorem 1 we choose m so that

$$2M \sum_{i=m+1}^\infty \overset{(+)}{\Delta} h_r^{(m)}(x_i) < \epsilon/4.$$

We find points $\{y_i\}$, $i = 1, 2, \dots, m$, such that y_i is between x_i and the nearest point of the set $\{x_i\}$ lying to the right of x_i and such that by the existence of $f(x_i + 0)$

$$|f(\xi_i) - f(x_i + 0)| < \frac{\epsilon}{4 \sum_{i=1}^m \overset{(+)}{\Delta} h_r^{(m)}(x_i)}$$

whenever $x_i < \xi_i \leq y_i$. Letting D_ϵ consist of the $\{x_i\}$ and $\{y_i\}$ together with the points a and b the proof follows almost exactly like (A) above. We have also the additional result:

$$\sigma^{(*)} \int_a^b f dh_r = \sum_{i=1}^\infty f(x_i + 0) \overset{(+)}{\Delta} h_r(x_i)$$

where x_i runs over the points of right discontinuity of h_r (or of g).

(C). *The case of $\sigma^{(*)} \int_a^b f dg_c$.*

The conditions are sufficient by a theorem of BLISS (see p. 1124) for the

existence of the ordinary integral in the norm sense or $N \int_a^b f dg$. This of course is sufficient for the existence of $\sigma^{(*)} \int_a^b f dg$.

The Modified Integral of Dushnik in the Norm Sense

Theorem 6. *In the case of f bounded, g of bounded variation in $\langle a, b \rangle$ for the existence of the integral $N^{(*)} \int_a^b f dg$ the following conditions are necessary and sufficient:*

(a^{**}). *In each point of the interval (a, b) and the right side of a and left side of b either f or g is continuous except that both may have removable discontinuity at the same point.*

(b^{**}). *Same as (b'') and (b^*).*

Proof. The following was given by GETCHELL as the condition of pseudo-additivity:

Condition (r). f and g have no points of common discontinuity in (a, b) and the right side of a and left side of b except that both may have removable discontinuity at the same point.

Since (b^{**}) is the same as (b^*) of theorem 5, in order to fulfill the conditions of Getchell's theorem we must show that together (a^*) and (r) are equivalent to (a^{**}).

We consider separately the set N_1 of points of (a, b) and the right side of a and left side of b in which either f or g is continuous and the set N_2 consisting of points for which both functions are discontinuous. For N_1 the three conditions (a^*), (a^{**}), and (r) hold trivially.

For a point of N_2 suppose that (a^*) and (r) both hold. Then (a^{**}) holds since it is the same as condition (r). On the other hand if (a^{**}) holds then (r) holds, being the same. Also the sidewise limits required by (a^*) are guaranteed. Hence (a^*) holds for the set. This completes the proof.

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Die Lektüre der in diesen Proceedings erschienenen Arbeit von DENISTON führte mich zu nachfolgenden Betrachtungen. Sie zeigen (siehe insbes. § 5) zu welchen Vereinfachungen der Gebrauch von ε -Teilungen eines Intervalls in bezug auf eine zugehörige Funktion von beschränkter Variation (Def. 3) in den Definitionen der Stieltjesschen Integrale, gegenüber dem Gebrauch der gewöhnlichen Teilungen, führen kann.

§ 1. **Definition 1.** $\alpha(x)$ sei von beschränkter Variation in $[a, b]$. Ein Punkt x von $[a, b]$ heisse *Doppelpunkt in bezug auf $\alpha(x)$* , wenn $\alpha(x)$ unstetig ist in x ; wir denken uns zwei *uneigentliche* Punkte $\overline{x-0}$ und $\overline{x+0}$ mit x zusammengefallen, und definieren α in diesen Punkten wie folgt:

$$\alpha(\overline{x-0}) = \alpha(x-0) \quad \text{oder} \quad \lim_{x' \rightarrow x; x' < x} \alpha(x'), \text{ und}$$

$$\alpha(\overline{x+0}) = \alpha(x+0) \quad \text{oder} \quad \lim_{x' \rightarrow x; x' > x} \alpha(x'). \quad ^1)$$

Wir schreiben $\xi_1 < \overline{x-0} < x < \overline{x+0} < \xi_2$, wenn $\xi_1 < x < \xi_2$ ist; die Axiome der linearen Anordnung behalten dadurch ihre Gültigkeit im neuen Punktbereich, gebildet von den Punkten der Zahlengeraden und den zu den Doppelpunkten (x) von $\alpha(x)$ gehörenden Punkten $(\overline{x-0})$, $(\overline{x+0})$.

Definition 2. $\alpha(x)$ sei in $[a, b]$ von beschränkter Variation; $f(x)$ sei daselbst endlich. In den zu den Doppelpunkten (x) in bezug auf $\alpha(x)$ gehörenden uneigentlichen Punkten $(\overline{x-0})$, $(\overline{x+0})$ definieren wir:

$$f(\overline{x-0}) = f(\overline{x+0}) = f(x).$$

Definition 3. Bei willkürlich positivem ε wird

$$\overline{a-0} \text{ oder } a = x_0 < x_1 < \dots < x_j < \dots < x_n = b \text{ oder } \overline{b+0} .$$

eine ε -Teilung von $[a, b]$ in bezug auf $\alpha(x)$ sein, wenn: 1° die Teilungspunkte entweder eigentliche oder (zu einem Doppelpunkt von $\alpha(x)$ gehörende) Punkte $(\overline{x-0})$ oder $(\overline{x+0})$ sind, 2° zu jedem Doppelpunkt x von $\alpha(x)$ mit

$$V(\overline{x+0}) - V(\overline{x-0}) \geq \varepsilon,$$

wobei $V(x)$ die Totalvariation von $\alpha(x)$ über $[a, x]$ für $a < x \leq b$ darstellt, und $V(a) = 0$ ist, unter den Teilungspunkten drei aufeinander folgende,

¹⁾ Wenn a oder b Unstetigkeitspunkt von $\alpha(x)$ ist, sei $\alpha(\overline{a-0}) = \alpha(a)$ bzw. $\alpha(\overline{b+0}) = \alpha(b)$.

x_k, x_{k+1} , und x_{k+2} , zu finden sind mit $x_k = \overline{x-0}$, $x_{k+1} = x$ und $x_{k+2} = \overline{x+0}$,
 3° für jedes Paar aufeinander folgender Teilungspunkte (x_l, x_{l+1}) , welches
 nicht zu den unter 2° fallenden Paaren $(\overline{x-0}, x)$, $(x, \overline{x+0})$ gehört,

$$V(x_{l+1}) - V(x_l) < \varepsilon$$

ist.

Definition 4. Eine linksseitige [rechtsseitige] ε -Summe von $f(x)$ in
 $[a, b]$ in bezug auf $a(x)$ hat die Gestalt

$$\sum_{j=1}^n f(x_{j-1}) \cdot \{a(x_j) - a(x_{j-1})\} \quad (1)$$

[bzw.

$$\sum_{j=1}^n f(x_j) \cdot \{a(x_j) - a(x_{j-1})\}; \quad (1^{\text{bis}})$$

dabei sollen die Teilungspunkte $x_0 < x_1 < \dots < x_n$ eine ε -Teilung von
 $[a, b]$ in bezug auf $a(x)$ bilden.

Definition 5. In $[a, b]$ sei $a(x)$ von beschränkter Variation, $f(x)$ be-
 schränkt. Dann hat $f(x)$ über $[a, b]$ ein linksseitiges [rechtsseitiges]
 Moore-Smith'sches. Integral in bezug auf $a(x)$, wenn die in $[a, b]$ zu
 $f(x)$ gehörenden linksseitigen [bzw. rechtsseitigen] ε -Summen in bezug
 auf $a(x)$ einen Moore-Smith'schen Grenzwert T [bzw. U] haben ²⁾; wir
 schreiben dann

$$T = (MS) \int_a^b f(x) da(x)$$

[bzw.

$$U = (MS) \int_a^b f(x) da(x)].$$

Satz 1. Zur Existenz des Integrales

$$(MS) \int_a^b f(x) da(x)$$

ist notwendig und hinreichend, dass die linksseitigen Unstetigkeitspunkte
 von $f(x)$, welche nicht gleichzeitig Unstetigkeitspunkte von $a(x)$ sind, eine
 Menge bilden mit $V(x)$ -Mass (L) Null; dabei ist $V(a) = 0$, $V(x) =$
 Totalvariation von $a(x)$ über $[a, x]$ für $a < x \leq b$.

Der Beweis verläuft wie der des Theorems 1 in DENISTON, *Existence of
 Stieltjes Integrals* ³⁾.

²⁾ D.h. zu willkürlich positivem η gibt es ein positives ε und dabei eine ε -Teilung von
 $[a, b]$ in bezug auf $a(x)$ mit den Eigenschaften: 1° die zugehörige Summe (1) und T
 [bzw. U] haben eine Differenz, welche absolut genommen $< \eta$ ist, 2° dasselbe gilt für
 jede linksseitige [bzw. rechtsseitige] ε -Summe, welche aus der vorigen durch Hinzufü-
 gung neuer Teilungspunkte zu den (x_j) hervorgeht.

³⁾ Cf. H. SCHÄRF, *Portugaliae Mathematica* 4, 90—92 (1943/45).

Satz 1bis. Zur Existenz von

$$(MS) \int_a^b f(x) da(x)$$

ist notwendig und hinreichend, dass die rechtsseitigen Unstetigkeitspunkte von $f(x)$, welche nicht gleichzeitig Unstetigkeitspunkte von $a(x)$ sind, eine Menge bilden mit $V(x)$ -Masse (L) Null ($V(x)$ definiert wie in Satz 1).

§ 2. **Definition 6.** $a(x)$ sei in $[a, b]$ von beschränkter Variation, $f(x)$ sei daselbst beschränkt. Dann hat $f(x)$ über $[a, b]$ ein linksseitiges [rechtsseitiges] normales Integral in bezug auf $a(x)$, wenn die in $[a, b]$ zu $f(x)$ gehörenden linksseitigen [bzw. rechtsseitigen] ε -Summen in bezug auf $a(x)$ für $\varepsilon \rightarrow 0$ einen Grenzwert V [bzw. W] haben; wir schreiben dann

$$V = (N) \int_a^b f(x) da(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

[bzw.

$$W = (N) \int_a^b f(x) da(x)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2^{bis})$$

Satz 2 [Satz 2bis]. Zur Existenz von

$$(N) \int_a^b f(x) da(x)$$

[von

$$(N) \int_a^b f(x) da(x)]$$

ist notwendig und hinreichend, dass die linksseitigen [bzw. rechtsseitigen] Unstetigkeitspunkte von $f(x)$, welche nicht gleichzeitig Unstetigkeitspunkte von $a(x)$ sind, eine Menge bilden mit $V(x)$ -Mass (L) Null ($V(x)$ definiert wie in Satz 1).

Der Beweis verläuft wie der des Satzes 1 in SCHÄRF, Port. Math. 4 (1943/45), S. 86—92⁴⁾.

§ 3. **Definition 7.** Eine ε -Summe von $f(x)$ in $[a, b]$ in bezug auf $a(x)$ hat die Gestalt

$$\sum_{j=1}^n f(\xi_{j-1}) \cdot \{a(x_j) - a(x_{j-1})\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

wobei die x_j eine ε -Teilung von $[a, b]$ in bezug auf $a(x)$ bilden, und für jedes zugelassene j : $x_{j-1} \leq \xi_{j-1} \leq x_j$ ist.

⁴⁾ Die Notwendigkeit der Bedingung ist auch eine unmittelbare Folge von Satz 1 [Satz 1 bis].

Definition 8. $a(x)$ sei in $[a, b]$ von beschränkter Variation, $f(x)$ beschränkt. Dann hat $f(x)$ über $[a, b]$ ein *normales Integral in bezug auf* $a(x)$, wenn die Summen (3), unabhängig von der Wahl der Punkte (ξ_{j-1}) in den zugehörigen Intervallen $[x_{j-1}, x_j]$, für $\varepsilon \rightarrow 0$ einen Grenzwert G haben; wir schreiben dann

$$G = (N) \int_a^b f(x) da(x).$$

Definition 9. Unter den gleichen Annahmen für $a(x)$ und $f(x)$ wie in Definition 8 hat $f(x)$ über $[a, b]$ ein *Moore-Smith'sches Integral in bezug auf* $a(x)$, wenn die Summen (3), unabhängig von der Wahl der Punkte (ξ_{j-1}) in den zugehörigen Intervallen $[x_{j-1}, x_j]$, einen Moore-Smith'schen Grenzwert H ⁵⁾ haben; wir schreiben dann

$$H = (MS) \int_a^b f(x) da(x).$$

Satz 3. Zur Existenz von

$$(N) \int_a^b f(x) da(x)$$

ist notwendig und hinreichend, dass die Unstetigkeitspunkte von $f(x)$, welche nicht gleichzeitig Unstetigkeitspunkte von $a(x)$ sind, eine Menge bilden mit $V(x)$ -Mass (L) Null ($V(x)$ definiert wie in Satz 1).

Einen Beweis findet man in unserer Arbeit in *Prace mat.-fiz.* **41** (1933), S. 74, 75 (Beweis von Satz V) ⁶⁾.

Satz 3bis. Auch zur Existenz von

$$(MS) \int_a^b f(x) da(x)$$

ist die in Satz 3 gegebene Bedingung notwendig und hinreichend.

Dies folgt sofort aus Satz 3 und *Prace mat.-fiz.* **41** (1933), S. 69 (Satz 1).

§ 4. **Definition 10** und **Definition 11** des *normalen Dushnik-Integrals* bzw. des *Moore-Smith-Dushnik-Integrals* von $f(x)$ über $[a, b]$ in bezug auf $a(x)$ sollen aus Def. 8 bzw. Def. 9 dadurch hervorgehen, dass man nur ε -Summen (3) zulässt mit $x_{j-1} < \xi_{j-1} < x_j$ für alle Paare (x_{j-1}, x_j) , welche sich nicht schreiben lassen in einer der Formen $(x, \overline{x+0})$, $(\overline{x-0}, x)$ ⁷⁾.

⁵⁾ Die Definition des Grenzwertes liegt auf der Hand; vgl. Fussn. 2.

⁶⁾ Aus der loc. cit. gegebenen Lebesgue-Radonschen Form der Definition des normalen Integrals folgt, dass jedes derartige Integral von $f(x)$ in bezug auf $a(x)$ über $[a, b]$ den gleichen Wert hat wie das dann auch existierende Lebesgue-Stieltjessche Integral von $f(x)$ über $[a, b]$ in bezug auf das aus $a(x)$ in bekannter Weise hervorgehende Lebesguesche Mass $m_{a(x)}(E)$.

⁷⁾ Wir nennen sie *Dushnik-Summen*.

Sind die zugehörigen Grenzwerte V_1 bzw. W_1 , so schreiben wir

$$V_1 = (ND) \int_a^b f(x) da(x)$$

bzw.

$$W_1 = (MSD) \int_a^b f(x) da(x).$$

Satz 4. Zur Existenz von

$$(MSD) \int_a^b f(x) da(x)$$

ist die in Satz 3 gegebene Bedingung notwendig und hinreichend.

Der Beweis verläuft wie der des Theorems 5 in der zitierten Arbeit von DENISTON⁸⁾. Ein zweiter Beweis der Notwendigkeit der Bedingung folgt sofort aus einer Bemerkung in Fussn. 9 (S. 70, 71) unserer Arbeit in *Prace mat.-fiz.* 41 (1933), welche besagt, dass die loc. cit. betrachteten oberen und unteren Integrale den gleichen Wert beibehalten bei Benutzung offener statt abgeschlossener Teilintervalle in den Definitionen der oberen und unteren ε -Summen.

Satz 4bis. Auch zur Existenz von

$$(ND) \int_a^b f(x) da(x)$$

ist die in Satz 3 gegebene Bedingung notwendig und hinreichend.

Die Notwendigkeit folgt unmittelbar aus Satz 4; dass die Bedingung hinreicht, ist eine Folge von Satz 3.

§ 5. Da für jede in $[a, b]$ definierte Funktion $f(x)$ in jedem Punkte x von $[a, b] - E$, wobei E eine abzählbare Teilmenge von $[a, b]$,

$$\limsup_{\xi \rightarrow x; \xi < x} f(\xi) = \limsup_{\xi \rightarrow x; \xi > x} f(\xi)$$

und

$$\liminf_{\xi \rightarrow x; \xi < x} f(\xi) = \liminf_{\xi \rightarrow x; \xi > x} f(\xi)$$

ist⁹⁾, sind die in den Sätzen 1, 1bis, 2, 2bis enthaltenen notwendigen und hinreichenden Bedingungen äquivalent mit den in den Sätzen 3, 3bis, 4, 4bis enthaltenen.

Somit führen die Definitionen der links- und rechtsseitigen normalen oder Moore-Smith'schen Integrale, die der normalen und der Moore-Smith'schen Integrale und die der normalen Dushnik- und der MS-Dushnik-Integrale genau ebenso weit.

⁸⁾ Dass die Bedingung hinreichend ist, folgt natürlich auch mit Satz 3 bis.

⁹⁾ Siehe E. W. HOBSON, *Theory of functions I* (Third Ed. 1927), p. 304, oder O. HAUPT, *Differential- und Integralrechnung I* (1938), S. 150.

D.h.: bei Benutzung von ε -Teilungen von $[a, b]$ in bezug auf $a(x)$ führen linksseitige und rechtsseitige, gewöhnliche und Dushnik-Summen sowohl unter Anwendung des normalen wie eines Moore-Smith'schen Grenzwertes zu demselben Integralbegriff. Der Integralwert fällt dabei mit dem des (sodann auch existierenden) Lebesgue-Stieltjesschen Integrals in bezug auf die zugehörige totaladditive Massfunktion $m_{a(x)}(E)$ zusammen.

Dagegen zeigt die Arbeit von DENISTON, dass bei Benutzung der üblichen Zerlegungen von $[a, b]$ Integralbegriffe von ungleichem Umfange erhalten werden ¹⁰⁾.

¹⁰⁾ Die notwendigen und hinreichenden Bedingungen in den Theoremen von DENISTON lassen sich unter Anwendung seiner Terminologie, auf folgende übersichtliche Weise zusammenfassen: 1^o für Theorem 1: (α) in jedem Punkte x von $(a, b]$, in welchem $g(x)$ linksseitig unstetig ist, existiert der linksseitige Grenzwert $f(x-0)$, (β) die Unstetigkeitspunkte von $f(x)$, welche nicht gleichzeitig Unstetigkeitspunkte von $g(x)$ sind, bilden eine Nullmenge in bezug auf $g(x)$; 2^o für Theorem 3: (α') in jedem Punkte x von $(a, b]$, in welchem $g(x)$ linksseitig unstetig ist, existiert der linksseitige Grenzwert $f(x-0)$, (α'') in jedem Punkte x von (a, b) , in welchem $g(x)$ rechtsseitig unstetig ist, ist $f(x)$ linksseitig stetig, (β) wie oben; 3^o für Theorem 4: (α') in jedem Punkte x von $(a, b]$, in welchem $g(x)$ linksseitig unstetig ist, ist $f(x)$ linksseitig stetig, (α'') in jedem Punkte x von $[a, b]$, in welchem $g(x)$ rechtsseitig unstetig ist, ist $f(x)$ rechtsseitig stetig, (β) wie oben; 4^o für den Satz (von BLISS), loc. cit. „Remark on the ordinary Stieltjes integral“: (α) in den Unstetigkeitspunkten von $g(x)$ ist $f(x)$ stetig, (β) wie oben; 5^o für Theorem 5: (α') in jedem Punkte x von $(a, b]$ in welchem $g(x)$ linksseitig unstetig ist, existiert $f(x-0)$, (α'') in jedem Punkte x von $[a, b]$; in welchem $g(x)$ rechtsseitig unstetig ist, existiert $f(x+0)$, (β) wie oben; 6^o für Theorem 6: (α) in den Unstetigkeitspunkten von $g(x)$ ist $f(x)$ stetig, ausgenommen vielleicht in Punkten, in welchen $g(x)$ eine hebbare Unstetigkeit hat; in einem derartigen Punkte soll dann auch $f(x)$ nur eine hebbare Unstetigkeit haben; (β) wie oben.

Die hier ausgeführte Umbildung der DENISTON'schen Bedingungen folgt unschwer unter Anwendung eines der in Fussn. 9 zitierten Sätze.

Mathematics. — *Line geometry and Quantum mechanics.* By E. M. BRUINS.
(Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of November 26, 1949.)

§ 1. *Introduction.*

The essential difference between classical physics and quantum physics lies in the fact, that classical physics specifies a physical state by a *finite* number of coordinates, whereas quantum physics uses in principle an infinite number of coordinates. The series of coordinates e.g.

$$a_1, a_2, a_3, \dots, a_n, \dots$$

was represented in quantum physics (using some system of orthogonal functions) by a function of an auxiliary variable ξ , the wave function. Transition to a different ordertype of the coordinates gives us another representation of the same state: e.g. the ordertype ω 2

$$a_1, a_3, a_5, \dots ; a_2, a_4, a_6, \dots$$

specifies the states by means of *two* functions of one variable. An ordertype ω^n . κ would specify the states by means of κ functions of n variables. So neither the number of functions, nor the number of variables used is of *fundamental importance*.

On the other hand the structure of Dirac equations, being formally that of the incidence of a point and a line in threedimensional projective space G_4 , suggests from the mathematical point of view the use of the theory of projective invariants, of complex symbols rather than a tensor calculus using special metrical tensors. Also the Poisson brackets of classical physics correspond to algebraical expressions in quantum mechanics! In the following we show that indeed the formalism of a geometry of second order, that of line geometry in G_4 is extremely simple used for quantum-physical purposes.

§ 2. *Generalisation of Dirac equations.*

a. A linecomplex q^2 in G_4 :

$$q_{12} \pi_{34} + q_{13} \pi_{42} + \dots = 0,$$

can be represented symbolically by $(q^2 \pi^2) = 0$ and has only one invariant

$$(q^2 q_1^2) = \Omega(q^2) = q_{12} q_{34} + q_{13} q_{42} + q_{14} q_{23},$$

from which, as is well known, follows an isomorphism of the projective group in G_4 and the sexternary group leaving invariant $\Omega(q^2)$. The relation

can be transformed by one of FELIX KLEIN's substitutions in the sum of six squares e.g.

$$\begin{aligned} Q_4 + iQ_6 &= q_{12} \quad , \quad Q_1 - iQ_2 = q_{13} \quad , \quad -Q_3 - iQ_5 = q_{14} \\ Q_4 - iQ_6 &= q_{34} \quad , \quad Q_1 + iQ_2 = q_{42} \quad , \quad -Q_3 + iQ_5 = q_{23}. \end{aligned}$$

With these coordinates corresponds a differential operator in six-dimensional Euclidean space E_6 :

$$\Delta_4 + i\Delta'_6 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial i x_6} = \partial'_{12} = -\partial'_{21} = \partial_{34} = -\partial_{43}, \text{ etc.}$$

which allows to introduce an alternating differential operator of order $\frac{1}{2}$ in threedimensional space described by complex symbols transforming cogredient to the points. We have then immediately the relations between the sixdimensional linear form $(\Delta'Q)$ and the quarternary bracket $(\partial^2 q^2)$

$$\begin{aligned} (\Delta'Q) &\equiv 2 \operatorname{div} Q \equiv (\partial^2 q^2) \\ (\partial^2 \partial_1^2) &\equiv \Delta_6 \equiv \text{Laplaceoperator.} \end{aligned}$$

An alternating 2-tensor in E_6 corresponds with a pencilcomplex in G_4 ; a symmetrical 2-tensor with a quadratic linecomplex.

Rot P' can e.g. be represented by

$$(\Delta'\pi)(\pi P') \text{ in } E_6, \quad \pi_i \pi_k = -\pi_k \pi_i,$$

and pushing down into G_4 we obtain

$$(\partial^2 \pi^2)(\varrho^2 p^2) \quad , \quad \pi^2 \cdot \varrho^2 = -\varrho^2 \cdot \pi^2.$$

The components of Rot P' are thus expressed by linear combinations of the 15 coefficients of a pencilcomplex in G_4 .

The symbolical methods for the study of these forms have already been discussed by R. WEITZENBÖCK, *Komplexsymbolik* (1908).

b. The necessary and sufficient condition for the possibility of splitting up

$$q_{ik} = (fg)_{ik} = \begin{vmatrix} f_i & g_i \\ f_k & g_k \end{vmatrix},$$

in which f, g are transforming cogredient to the points of G_4 is, that the linecomplex q^2 be a special linecomplex i.e. $(q^2 q_1^2) = 0$.

The Moebius correlation with regard to a general linecomplex is given by $(q^2 xy) = 0$. There exist points with an undetermined conjugated plane only for special complexes and for the points x , on the axis of the complex, holds

$$(q^2 xy) \equiv 0 \{y\}.$$

The equations of motion of the free Dirac electrons are given by the condition that f lies on the axis of the special linecomplex ∂^2 .

The conditions

$$(\partial^2 fg) \equiv 0 \{g\}$$

read, written in full

$$\left. \begin{aligned} \partial_{34} f_2 + \partial_{42} f_3 + \partial_{23} f_4 &= 0 \\ \partial_{34} f_1 + \partial_{13} f_4 + \partial_{41} f_3 &= 0 \\ \partial_{21} f_4 + \partial_{42} f_1 + \partial_{14} f_2 &= 0 \\ \partial_{12} f_3 + \partial_{23} f_1 + \partial_{31} f_2 &= 0 \end{aligned} \right\}$$

Introducing the operators

$$p_k = \frac{\partial}{\partial x_k}$$

and the matrices

$$\begin{aligned} a_1 &= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, & a_2 &= -i \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, & a_3 &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \\ a_4 &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, & a_5 &= i \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, & a_6 &= -i \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \end{aligned}$$

this system reads

$$(\sum_k a_k p_k) \cdot f = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

A quantity $f_i \neq 0$ can only exist if the symbolical invariant of the operator ∂^2 vanishes i.e.

$$(\partial^2 \partial_1^2) \cdot f \equiv \Delta_6 \cdot f = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From the fact that the p_k transform contragredient to the points of E_6 and from the invariance of equation (1) it follows that the a_k must transform cogredient to the points of E_6 . (see also Remark I).

To a physical state we make correspond a special symbolical linecomplex. The equations (1), (2) then contain a generalisation of the Dirac equations of motion of the free electron, and are easily identified with those if we, e.g., specialising, put

$$f_i = e^{Ax_i} f_i^* (x_1, x_2, x_3, x_6).$$

c. General equations of motion for non-free particles lie before hand: they can be obtained from the, only possible, projective invariant form

$$(\partial^2 f g) + (c^2 f g) \equiv 0 \{g\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If we put, specialising,

$$\frac{\partial f_i}{\partial x_4} = (m_0 c - A) f_i, \quad \frac{\partial f_i}{\partial x_5} = S f_i$$

and apart from numerical constants

$$C = (\alpha_x, \alpha_y, \alpha_z, A, S, i\varphi)$$

we obtain the well known Dirac equations for the electrons in an arbitrary field. The postulate of Dirac

$$p_k \rightarrow p_k + c_k$$

corresponds to the postulate of projective invariance in G_4 .

That the corresponding wave equation of the general equations of motion contains 15 additional spin terms is evident from the multiplication into $(\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 p_4 + \alpha_5 p_5 - \alpha_6 p_6)$ of (3).

d. We can also specialise the equations (3) by supposing a relation between the f_i , which can be interpreted as a *structure of the particle*. If e.g. we put

$$\frac{\partial f_n}{\partial x_6} = \sum_k A_{nk} f_k$$

the equation (3) becomes with 5 variables

$$(\sum \alpha_\mu \frac{\partial}{\partial x_\mu} + \alpha_\mu L_\mu + \alpha_\mu \alpha_\nu A_{\mu\nu} + M \|1\|) \cdot f = 0$$

$$A_{\mu\nu} = -A_{\nu\mu} \quad (\mu, \nu = 1, 2, 3, 4, 5)$$

which are *Møller's meson equations*¹⁾. The matrix $\|A\|$ can be a general matrix; without a relation between the f_i we cannot obtain terms $\alpha_\mu \alpha_\nu$ in an invariant way as a matrix that anticommutes with $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ must vanish.

§ 3. Densities in quantum mechanics.

a. Let us consider the quantities

$$j_k = \varphi' a_k \varphi,$$

the sixth component is apart from a constant the quaternary linear form $I = (\varphi' \varphi)$. As is easily controlled we have

$$\sum_{k=1}^6 j_k^2 = \sum_{k=1}^6 (\varphi' a_k \varphi)^2 = 0.$$

The vector j_k in E_6 is an isotrope vector.

b. Between the a_k , $k = 1, 2, 3, 4, 5$ we have the relations

$$a_i a_k = -a_k a_i$$

$$a_1 a_2 a_3 a_4 = a_5.$$

If now we form the quantities

$$P_k = \varphi' a_k \varphi ; p_{ik} = \varphi' a_i a_k \varphi ; q_{ikl} = \varphi' a_i a_k a_l \varphi ; a'_a = \varphi' a_i a_k a_l a_m \varphi$$

¹⁾ C. MØLLER, D. Danske Vid. Selsk. math.-fys. Medd., 18 (1941).

we can represent P, p^2, q^3, a' by a point P , a planecomplex p^2 , a linecomplex q^3 and a G_4 in G_5 .

Because of the commutationrules between the α_k we have

$$\begin{aligned} \alpha_l \alpha_k \alpha_l &= -\text{signum}(i k l m n) \alpha_m \alpha_n \\ \alpha_l \alpha_k \alpha_l \alpha_m &= +\text{signum}(i k l m n) \alpha_n \end{aligned}$$

or

$$P_k = a'_k \quad p_{ik} = -q'_{ik},$$

q'_{ik} being the space coordinates of q^3 . If there did not exist relations of the form

$$(p_1^2 p_2^2 x) = \text{factor}(a' x) : (p^2 P \pi^2) = \text{factor}(q^3 \pi^2)$$

the physical system would apart from the linearform $(\varphi' \varphi)$ have other invariants, independent of this one, as $(p^2 p_1^2 P)$, $(p^2 q^3)$, ...

We have to control the projective invariant relations between P, p^2, q^3, a' .

I. We consider the linecomplex $(p^2 P \pi^2)$ and calculating the 15 relations we obtain

$$\sum_{\text{cycl } i, k, l} p_{ik} P_l = I q_{ikl}$$

or

$$(p^2 P \pi^2) \equiv I(q^3 \pi^2) \quad \{\pi^2\}.$$

II. The space $(p^2 p_1^2 x)$ leads to calculate the 5 forms

$$p_{ik} p_{lm} + p_{il} p_{mk} + p_{im} p_{lk} = I a'_n$$

or

$$(p^2 p_1^2 x) \equiv I(a' x) \quad \{x\}.$$

So there is only one invariant I , $(a'P) = I^2$ and the relations between the "densities" are in geometrical form

I. q^3 is the compound of p^2 and P .

II. a' is the focalspace of the planecomplex p^2 ;

P is the focus of the linecomplex q^3 .

c. In sixdimensional space we can build a G_4 -complex with

$$\begin{aligned} p_{ik} &= \varphi' \alpha_l \alpha_k \varphi & (i, k = 1, 2, 3, 4, 5) \\ p_{k6} &= -p_{6k} = \varphi' \alpha_k \varphi \end{aligned}$$

We, then, have firstly to consider the linecomplex

$$(p^2 p_1^2 \pi^2) = (s^4 \pi^2).$$

From b we see immediately

$$s_{iklm} = (p^2 p_1^2)_{iklm} = I P_n = I p_{n6}$$

$$s_{ikl6} = p_{ik} p_{l6} + p_{il} p_{6k} + p_{i6} p_{kl} = p_{ik} P_l + p_{li} P_k + p_{kl} P_i = I q_{ikl} = I p_{mn}.$$

We can therefore sum up all relations given by I, II in

$$\underline{(p^2 p_1^2 \pi^2) \equiv I (r^4 \pi^2) \quad (\pi^2)}$$

$$r'_{ik} = p_{ik} \quad i, k = 1, 2, 3, 4, 5, 6.$$

The invariant of the complex is now evidently

$$(p^2 p_1^2 p_2^2) \equiv I \Sigma p_{ik}^2 \equiv 2I^3 \quad \text{as} \quad \Sigma p_{ik}^2 = (p^2 q^3) + I^2.$$

Remark I.

If we transform (f_1, f_2, f_3, f_4) by a general projective transformation S of G_4

$$\bar{f} = S \cdot f$$

the bracket $(\partial^2 f g)$ will be transformed, apart from the transformation modulus, the determinant of S , in $(\bar{\partial}^2 \bar{f} \bar{g})$. The generalised Dirac equations will then be transformed in

$$(\Sigma a_k \bar{p}_k) \bar{f} = (\Sigma a_k \bar{p}_k) \cdot S f.$$

As we remarked already in § 2 the fact that p_k transform contragredient to the points in E_6 in the orthogonal transformation corresponding to the projective transformation S in G_4 we can multiply at the left side by an operator T restore the original equations whilst

$$f_i(x_1, \dots, x_6) = \bar{f}_i(\bar{x}_1, \dots, \bar{x}_6).$$

The a_k are then transformed cogredient to the points of E_6 . We will deduce the explicit form of T .

a. The general projective transformation S of G_4 can be represented by the symbolical product²⁾

$$S: (a' x) (a u') \quad \text{which reads} \quad \tilde{x}_i = \sum_k a_i a'_k x_k.$$

The determinant of the transformation is $D = (a' b' c' d') (\alpha \beta \gamma \delta)$, apart from a constant. The inverse transformation is

$$S^{-1}: (a' b' c' u') (\alpha \beta \gamma x) = (a' x) (a u')$$

Indeed

$$S^{-1} S: (a' x) (a a') (a u') = \frac{1}{4} D \cdot (u' x)$$

as we obtain splitting up $a \quad a'$ in $(b' c' d')$, $(\beta \gamma \delta)$ and transforming the a' into the last bracket.

The transformation of the line coordinates in G can be easily found: intersecting

$$(a' x) (a u') = 0 \quad (b' y) (\beta u') = 0, \quad (x y)_{ik} = p_{ik}$$

²⁾ The projective invariants of S are given by the chains, the first of which is the diagonal sum

$$I_1 = (a' a), \quad I_2 = (a' \beta) (b' a), \quad I_3 = (a' \beta) (b' \gamma) (c' a), \quad I_4 = (a' \beta) (b' \gamma) (c' \delta) (d' a).$$

we have

$$(a' x) (b' y) (a \beta \pi^2) \equiv \frac{1}{2} (a' p) (b' p) (a \beta \pi^2)$$

or

$$\tilde{p}_{ik} = \{ (a \beta)_{ik} \sum (a' b')_{\lambda \mu} p_{\lambda \mu} = (a \beta)_{ik} L$$

where, introducing the sixdimensional coordinates $X_1, X_2, \dots, X_6$ we have

$$L = \{ (a' b')_{13} + (a' b')_{42} \} X_1 - i \{ (a' b')_{13} - (a' b')_{42} \} X_2 + \\ \{ -(a' b')_{14} - (a' b')_{23} \} X_3 - i \{ (a' b')_{14} - (a' b')_{23} \} X_5 + \\ \{ (a' b')_{12} + (a' b')_{34} \} X_3 + i \{ (a' b')_{12} - (a' b')_{34} \} X_6.$$

or calculating the $\tilde{X}_1, \dots, \tilde{X}_6$ from the $\tilde{p}_{ik}$

$$\tilde{X}_1 = \frac{(a \beta)_{13} + (a \beta)_{42}}{2} L, \quad \tilde{X}_2 = i \frac{(a \beta)_{13} - (a \beta)_{42}}{2} L, \quad \tilde{X}_3 = - \frac{(a \beta)_{14} + (a \beta)_{23}}{2} L \\ \tilde{X}_4 = \frac{(a \beta)_{12} + (a \beta)_{34}}{2} L, \quad \tilde{X}_5 = + i \frac{(a \beta)_{14} - (a \beta)_{23}}{2} L, \quad \tilde{X}_6 = - i \frac{(a \beta)_{12} - (a \beta)_{34}}{2} L.$$

As this transformation is an orthogonal transformation in E_6 we obtain the inverse transformation

$$X_i = \sum_k O_{ik} \tilde{X}_k$$

by simply interchanging the rows and columns, apart from a numerical constant.

On the other hand

$$\sum_k c_k a_k = \begin{vmatrix} c_4 - i c_6, & 0, & c_3 - i c_5, & c_1 - i c_2 \\ 0, & c_4 - i c_6, & c_1 + i c_2, & -c_3 - i c_5 \\ c_3 + i c_5, & c_1 - i c_2, & -c_4 - i c_6, & 0 \\ c_1 + i c_2, & -c_3 + i c_5, & 0, & -c_4 - i c_6 \end{vmatrix}$$

so we can easily express

$$\sum_k O_{ik} a_k = [a', b']_i \begin{vmatrix} (a \beta)_{34} & 0 & -(a \beta)_{14} & (a \beta)_{13} \\ 0 & (a \beta)_{34} & (a \beta)_{42} & (a \beta)_{23} \\ -(a \beta)_{23} & (a \beta)_{13} & -(a \beta)_{12} & 0 \\ (a \beta)_{42} & (a \beta)_{14} & 0 & -(a \beta)_{12} \end{vmatrix}$$

where $[a', b']_i$ is the symbolical coefficient of X_i in L .

Multiplying on the right with S we have as e.g.

$$(a \beta)_{34} \gamma_1 c'_i - (a \beta)_{24} \gamma_3 c'_i + (a \beta)_{23} \gamma_4 c'_i = (a \beta \gamma)_{341} c'_i$$

$$\sum_k O_{ik} a_k \cdot S = [a', b']_i \begin{vmatrix} \dots, (a \beta \gamma)_{341} c'_i, \dots \\ \dots, (a \beta \gamma)_{342} c'_i, \dots \\ \dots, (a \beta \gamma)_{231} c'_i, \dots \\ \dots, (a \beta \gamma)_{421} c'_i, \dots \end{vmatrix}$$

from which introducing the a, a' by using cyclical symmetry is obtained

$$\sum_k O_{ik} a_k \cdot S = T a_i$$

$$T = \begin{vmatrix} T_{22} & -T_{21} & -T_{24} & T_{23} \\ -T_{12} & T_{11} & T_{14} & -T_{13} \\ -T_{42} & T_{41} & T_{44} & -T_{43} \\ T_{32} & -T_{31} & -T_{34} & T_{33} \end{vmatrix} \quad T_{ik} = (S^{-1})_{ik}.$$

We have e.g. for $i = 1$ from

$$[(a' b')_{13} + (a' b')_{42}] c_1$$

$$\lambda = 1, (a' b' c')_{421}; \lambda = 2, (a' b' c')_{132}; \lambda = 3, (a' b' c')_{423}; \lambda = 4, (a' b' c')_{134}$$

$$\sum_k O_{1k} a_k \cdot S = \begin{vmatrix} a'_2 c_3 & -a'_2 c_4 & -a'_2 c_1 & a'_2 c_2 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{vmatrix} = T a_1.$$

Remark II.

The relations deduced in § 3 will be seen to correspond to the Pauli-identities. Recently G. PETIAU³⁾ has deduced these relations using tensor calculus, introducing a special metrical tensor. It may be of use to show the identity of his final results with the relations obtained above.

A. The relation

$$f_{\alpha\beta} \omega_1 = f_{\alpha\beta\gamma} j^\gamma$$

reads in projective notation

$$\omega_1 (p\pi')^2 \equiv (q\pi')^2 (qj') \equiv (q'^2 j' \pi'^2) \quad \{\pi\}$$

or

$$(q'^2 j')_{ikl} = \omega_1 p'_{ikl}.$$

B. The relation

$$f_{\alpha\beta\gamma} f^{\gamma\delta} = \omega_1 [g_\alpha^\delta j_\beta - g_\beta^\delta j_\alpha]$$

reads in projective notation

$$(q\pi')^2 (q q') (q' x) \equiv \omega_1 (g' g) (g' x) (g\pi') (j\pi')$$

which shows because of the fact that $(g' g)$ is a reductent on $(g' g)^2$ that this relation is completely independent of the metrical tensor used!! So it is superfluous to introduce a metrical tensor for these relations. We have

$$(q\pi')^2 (q q') (q' x) \equiv \text{const } \omega_1 (g' g)^2 (x\pi') (j\pi')$$

and transforming to pointcoordinates we have

$$(\pi^3 q^2) (q q_1^3 x) \sim (q^3 \pi^2) (\pi q_1^3 x) \sim -(\pi^2 q_1^2 x) (\pi q_1 q^3) \sim (q^3 q_1^2) (q_1 \pi^3 x)$$

or

$$(j u') \equiv (q^3 q_1^2) (q_1 u')$$

³⁾ Revue scientifique 83, 37 (1945).

which with the contravariant notation of q^3 results in

$$\omega_1(ju') \equiv (p'^2 p_1'^2 u').$$

Conclusion:

If we represent a physical state by a symbolical special linecomplex

$$p_{ik} = (fg)_{ik} \text{ in } G_4,$$

where f, g , are functions of six variables x_i , the generalised Dirac equations of motion correspond to the incidence of the point f with the axis of $\partial^2 + c^2$

$$(\partial^2 fg) + (c^2 fg) \equiv 0 \quad \{g\}$$

which is of the form

$$\sum_k a_k \left(\frac{\partial}{\partial x_k} + c_k \right) \cdot f$$

in which the spin matrices a_k transform like the point coordinates of the orthogonal transformations of x_i . We can use general projective transformations for the f . The MØLLER meson equations can be obtained by relations between the f_i . The relations between the "densities"

$$\varphi' a_i a_k \varphi \quad , \quad \varphi' a_k \varphi$$

can be considered as those of the coordinates of a G_4 -complex in $E_6 P_{ik}$, which, satisfies the relation

$$(P^2 P_1^2 \pi^2) \equiv (\varphi' \varphi) (R^4 \pi^2) \quad \{\pi^2\}$$

$$R'_{ik} = P_{ik}.$$

The Dirac postulate for the equations of motion in an external field follows automatically from the projective invariance in G_4 .

Mathematics. — *A cohomology theory with higher coboundary operators I, (Construction of the Groups *).* By SZE-TSEN HU. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of October 29, 1949.)

1. Introduction ¹⁾.

The purpose of the present work is to give a generalization of the classical cohomology theory in algebraic topology by introducing coboundary operators of higher order. In the present note, we shall construct such a theory in details by using WALLACE cochains of a topological space, as given in a recent paper of SPANIER, [2] ²⁾. However, one might easily establish analogous generalisations for the singular homology and cohomology theory and for the homology and cohomology groups of an abstract group.

2. Cochains of a space.

Let X be a topological space ³⁾. Following SPANIER, [2, p. 408], we shall denote by X^{m+1} the $(m+1)$ -fold topological product of X with itself and denote by ΔX^{m+1} the diagonal of X^{m+1} . Let G be a given discrete abelian group, written in the additive form.

By an m -function of X into G , we mean a function $\phi: X^{m+1} \rightarrow G$ defined on the space X^{m+1} with values in G . Continuity is not assumed. The totality of all m -functions of X into G form a group $\Phi^m(X, G)$ with functional addition as the group operation. An m -function ϕ of $\Phi^m(X, G)$ is said to be *locally zero* if there is an open set N of X^{m+1} containing ΔX^{m+1} such that ϕ is zero on N . It is easy to see that the set of all locally zero m -functions form a subgroup $\Phi_0^m(X, G)$ of $\Phi^m(X, G)$.

The factor group

$$C^m(X, G) = \Phi^m(X, G) / \Phi_0^m(X, G)$$

is defined to be the group of m -cochains of X over G , whose elements, the cosets of $\Phi_0^m(X, G)$ in $\Phi^m(X, G)$, will be called the m -cochains of X over G . We shall denote by $[\phi]$ the m -cochain which contains $\phi \in \Phi^m(X, G)$, i.e.

$$[\phi] = \phi + \Phi_0^m(X, G),$$

and ϕ will be called a *representative* of $[\phi]$.

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¹⁾ The author acknowledges his gratitude to Professor A. D. WALLACE for his kind suggestions.

²⁾ Numbers in brackets refer to the bibliography at the end of the paper.

³⁾ The topological spaces considered here are of the most general type, no separation axioms are postulated.

Further, we define $C^m(X, G) = 0$ if $m < 0$, i.e. the group which consists of a single element.

3. The p -th coboundary operator.

Let p be a given positive integer. For any given p integers $i_1, \dots, i_p$ such that

$$0 \leq i_1 < i_2 < \dots < i_p \leq m + p$$

we define a projection

$$\Pi_{i_1 \dots i_p} : X^{m+p+1} \rightarrow X^{m+1}$$

by taking

$$\Pi_{i_1 \dots i_p}(x_0, \dots, x_{m+p}) = (x_{j_0}, \dots, x_{j_m}),$$

where $(i_1, \dots, i_p, j_0, \dots, j_m)$ form a permutation of the $m + p + 1$ integers $0, 1, \dots, m + p$ with

$$j_0 < j_1 < \dots < j_m.$$

We shall define the p -th coboundary operator δ^p as follows. For an arbitrary m -function $\phi \in \Phi^m(X, G)$, its p -th coboundary $\delta^p \phi$ is an $(m + p)$ -function of $\Phi^{m+p}(X, G)$ defined by

$$\begin{aligned} (\delta^p \phi)(x_0, \dots, x_{m+p}) &= \\ &= \sum \operatorname{sgn} \begin{pmatrix} 0, \dots, p-1, p, \dots, m+p \\ i_1, \dots, i_p, j_0, \dots, j_m \end{pmatrix} \phi(\Pi_{i_1 \dots i_p}(x_0, \dots, x_{m+p})). \end{aligned}$$

The summation ranges over all the possible choices of the integers $i_1, \dots, i_p$ such that the integers $(i_1, \dots, i_p, j_0, \dots, j_m)$ form a permutation of the $m + p + 1$ integers $0, \dots, m + p$ satisfying the sectionwise normality condition:

$$i_1 < i_2 < \dots < i_p, \quad j_0 < j_1 < \dots < j_m.$$

For a given permutation P of objects, the value of $\operatorname{sgn}(P)$ is 1 or -1 according as P is even or odd.

The p -th coboundary operator δ^p , defined above, clearly is a homomorphism

$$\delta^p : \Phi^m(X, G) \rightarrow \Phi^{m+p}(X, G).$$

One of its important properties is:

$$\delta^p \text{ maps } \Phi_0^m(X, G) \text{ into } \Phi_0^{m+p}(X, G). \quad \dots \quad (3.1)$$

Proof. Let $\phi \in \Phi_0^m(X, G)$ be an arbitrary locally zero m -function. There is an open set N of X^{m+1} containing ΔX^{m+1} on which ϕ vanishes. Let $N^* \subset X^{m+p+1}$ denote the intersection of all sets $\Pi_{i_1 \dots i_p}^{-1}(N)$, then $\delta^p \phi$ is zero on N^* . Since $\Pi_{i_1 \dots i_p}$ is continuous and maps ΔX^{m+p+1} into ΔX^{m+1} , each $\Pi_{i_1 \dots i_p}^{-1}(N)$ is an open set containing ΔX^{m+p+1} . Since there are only a finite number of projections $\Pi_{i_1 \dots i_p}$ for a given m , N^* is an open set of X^{m+p+1} containing the diagonal ΔX^{m+p+1} . Hence we get $\delta^p \phi \in \Phi_0^{m+p}(X, G)$, Q.E.D.

It follows from (3.1) that the p -th coboundary operator δ^p , defined for m -functions, induces a homomorphism

$$\delta^p: C^m(X, G) \rightarrow C^{m+p}(X, G) \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

by the relation:

$$\delta^p[\phi] = [\delta^p \phi].$$

Further, we define δ^p in (3.2) to be $\delta^p = 0$ for each $m < 0$.

4. Successive coboundaries.

It is natural to ask whether or not the following equality

$$\delta^p \delta^q = 0$$

holds for a given pair of positive integers p and q . To answer this question, we shall calculate the $(m+p+q)$ -function $\delta^p \delta^q \phi$ for an arbitrary m -function $\phi \in \Phi^m(X, G)$.

By the definitions of δ^q and δ^p in § 3, one can easily deduce that

$$\begin{aligned} \delta^p \delta^q \phi(x_0, \dots, x_{m+p+q}) &= \\ &= \sum \operatorname{sgn} \begin{pmatrix} 0, \dots, p-1, p, \dots, p+q-1, p+q, \dots, m+p+q \\ i_1, \dots, i_p, j_1, \dots, j_q, k_0, \dots, k_m \end{pmatrix} \phi(x_{k_0}, \dots, x_{k_m}) \end{aligned}$$

where $i_1, \dots, i_p, j_1, \dots, j_q, k_0, \dots, k_m$ is any permutation of the $m+p+q+1$ integers $0, 1, \dots, m+p+q$ subjected to the condition that

$$i_1 < i_2 < \dots < i_p, \quad j_1 < j_2 < \dots < j_q, \quad k_0 < k_1 < \dots < k_m,$$

and the summation on the right member ranges over all of these permutations.

Let us consider a fixed choice of the integers $k_0, \dots, k_m$ such that

$$0 \equiv k_0 < k_1 < \dots < k_m \equiv m+p+q.$$

Let $h_1, \dots, h_{p+q}$ be $p+q$ integers such that $h_1 < h_2 < \dots < h_{p+q}$ and that $h_1, \dots, h_{p+q}, k_0, \dots, k_m$ form a permutation of the integers $0, 1, \dots, m+p+q$. The coefficient of $\phi(x_{k_0}, \dots, x_{k_m})$ in our formula for $\delta^p \delta^q \phi$ is

$$\begin{aligned} &\sum \operatorname{sgn} \begin{pmatrix} 0, \dots, p-1, p, \dots, p+q-1, p+q, \dots, m+p+q \\ i_1, \dots, i_p, j_1, \dots, j_q, k_0, \dots, k_m \end{pmatrix} = \\ &= \left\{ \sum \operatorname{sgn} \begin{pmatrix} h_1, \dots, h_p, h_{p+1}, \dots, h_{p+q} \\ i_1, \dots, i_p, j_1, \dots, j_q \end{pmatrix} \right\} \operatorname{sgn} \begin{pmatrix} 0, \dots, p+q-1, p+q, \dots, m+p+q \\ h_1, \dots, h_{p+q}, k_0, \dots, k_m \end{pmatrix} \end{aligned}$$

where $i_1, \dots, i_p, j_1, \dots, j_q$ is any permutation of the $p+q$ integers $h_1, \dots, h_{p+q}$ subjected to the condition of sectionwise normality:

$$i_1 < i_2 < \dots < i_p, \quad j_1 < j_2 < \dots < j_q,$$

and the summations on both sides are taken over all of these permutations.

Then it follows easily that

$$\delta^p \delta^q \phi = \theta(p, q) \delta^{p+q} \phi, \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

where $\theta(p, q)$ denotes the algebraic sum of the sign's of all (p, q) -section-wise normal permutations of $p + q$ integers $1, 2, \dots, p + q$. A permutation of $1, 2, \dots, p + q$ is said to be (p, q) -sectionwise normal, if it is of the form $i_1, \dots, i_p, j_1, \dots, j_q$ such that

$$i_1 < i_2 < \dots < i_p, \quad j_1 < j_2 < \dots < j_q.$$

5. The evaluation of $\theta(p, q)$.

To evaluate $\theta(p, q)$, it is easily verified that

$$\theta(p, q) = (-1)^{pq} \theta(q, p). \quad (5.1)$$

For an odd p , it follows immediately from (5.1) that $\theta(p, p) = 0$.

As a recurrence formula for calculation, we have

$$\theta(p, q) = \theta(p, q-1) + (-1)^q \theta(p-1, q), \quad (5.2)$$

whence one can deduce by induction that

$$\theta(p, q) = \frac{([p/2] + [q/2])!}{[p/2]! [q/2]!} \quad (5.3)$$

when pq is even, and $\theta(p, q) = 0$ when pq is odd. In (5.3), the symbol $[x]$ denotes the greatest integer not exceeding x and $0! = 1$. In particular, when p is even,

$$\theta(p, p) = p! / \{(p/2)!\}^2. \quad (5.4)$$

As a consequence of (5.3), we have

$$\theta(p, q) = \theta(q, p) \equiv 0 \quad (5.5)$$

for every pair of integers p and q . Hence,

$$\delta^p \delta^q \phi = \delta^q \delta^p \phi = \theta(p, q) \delta^{p+q} \phi$$

for every m -function $\phi \in \Phi^m(X, G)$, and the coboundary operators $\delta^p, \delta^q, \delta^{p+q}$ are related by

$$\delta^p \delta^q = \delta^q \delta^p = \theta(p, q) \delta^{p+q}. \quad (5.6)$$

6. The (p, q) -cohomology groups.

Throughout the remainder of the paper, let p, q be a given pair of positive integers and let

$$\theta = \theta(p, q) = \theta(q, p).$$

Let G be a discrete abelian group, written in the additive form. Assume that the order of every element of G is a divisor of θ , i.e. $\theta g = 0$ for each $g \in G$.

Consider the two coboundary operators

$$\delta^p : C^m(X, G) \rightarrow C^{m+p}(X, G),$$

$$\delta^q : C^{m-q}(X, G) \rightarrow C^m(X, G).$$

The cochains $c^m \in C^m(X, G)$ with $\delta^p c^m = 0$ are called *cocycles of order p* ; they form a subgroup

$$Z^{m,p}(X, G) \subset C^m(X, G).$$

which is the kernel of δ^p . The cochains $c^m \in C^m(X, G)$ such that $c^m = \delta^q c^{m-q}$ for some $c^{m-q} \in C^{m-q}(X, G)$ are called *coboundaries of order q* ; they form a subgroup

$$B^{m,q}(X, G) \subset C^m(X, G),$$

which is the image of δ^q .

It follows from our assumption concerning the order of the elements of G that $\delta^p \delta^q = 0$; hence

$$B^{m,q}(X, G) \subset Z^{m,p}(X, G).$$

The discrete factor group

$$H_{(p,q)}^m(X, G) = Z^{m,p}(X, G) / B^{m,q}(X, G)$$

will be called the m -dimensional (p, q) -cohomology group of X over G .

7. Induced homomorphisms of cochains.

Let $f: X \rightarrow Y$ be a (continuous) map of X into Y . f will also be used to denote the map $f: X^{m+1} \rightarrow Y^{m+1}$ defined by

$$f(x_0, \dots, x_m) = (f(x_0), \dots, f(x_m)).$$

The following properties are immediate:

$$f(\Delta X^{m+1}) \subset \Delta Y^{m+1}. \quad (7.1)$$

$$\Pi_{i_1 \dots i_p} f = f \Pi_{i_1 \dots i_p} \quad (7.2)$$

For each m -function $\phi \in \Phi^m(Y, G)$, let us define an m -function $f' \phi$ of $\Phi^m(X, G)$ by taking

$$f' \phi(x_0, \dots, x_m) = \phi(f(x_0), \dots, f(x_m)).$$

Then f' is a homomorphism

$$f': \Phi^m(Y, G) \rightarrow \Phi^m(X, G).$$

According to a statement of SPANIER, [2, p. 410], f' maps $\Phi_0^m(Y, G)$ into $\Phi_0^m(X, G)$. Hence, f' induces a homomorphism

$$f^\# : C^m(Y, G) \rightarrow C^m(X, G)$$

defined by $f^\#[\phi] = [f' \phi]$ for every chain $[\phi] \in C^m(Y, G)$.

The following properties of the homomorphism $f^\#$ are immediate:

$$\text{If } f: X \rightarrow X \text{ is the identity map, then } f^\# \text{ is the identity automorphism} \quad (7.3)$$

$$\text{If } f: X \rightarrow Y \text{ and } g: Y \rightarrow Z, \text{ then } (gf)^\# = f^\# g^\#. \quad (7.4)$$

$$f^\# \delta^p = \delta^p f^\#. \quad (7.5)$$

8. The relative groups.

A pair (X, X_0) is defined as a topological space X and a subspace ⁴⁾

⁴⁾ By a subspace of a topological space X , we mean any subset of X , not necessarily closed, with the topology obtained by relativisation.

X_0 . Let (X, X_0) be a given pair and $\iota: X_0 \rightarrow X$ be the identity map. Then the induced homomorphism

$$\iota^\# : C^m(X, G) \rightarrow C^m(X_0, G)$$

is defined. The kernel of this homomorphism $\iota^\#$ is called *the group of m -dimensional cochains of X modulo X_0 over G* , denoted by

$$C^m(X \bmod X_0, G).$$

As a consequence of (7.5), we have:

$$\delta^p C^m(X \bmod X_0, G) \subset C^{m+p}(X \bmod X_0, G). \quad (8.1)$$

Let

$$Z^{m,p}(X \bmod X_0, G) = C^m(X \bmod X_0, G) \cap Z^{m,p}(X, G),$$

$$B^{m,q}(X \bmod X_0, G) = \delta^q C^{m-q}(X \bmod X_0, G).$$

Then it follows from (8.1) and our assumption on G that

$$B^{m,q}(X \bmod X_0, G) \subset Z^{m,p}(X \bmod X_0, G).$$

The discrete factor group

$$H_{(p,q)}^m(X \bmod X_0, G) = Z^{m,p}(X \bmod X_0, G) / B^{m,q}(X \bmod X_0, G)$$

is defined to be the m -dimensional (p, q) -cohomology group of X modulo X_0 over G . If X_0 is vacuous, then it reduces to $H_{(p,q)}^m(X, G)$.

Appendix A. A modification of the theory.

In our definition of the (p, q) -cohomology groups over G , given in §§ 6 and 8, we assume that $\theta(p, q)g = 0$ for every element g of G . To get rid of this undesirable assumption, a modification of the theory was proposed to the author by Prof. A. D. WALLACE as follows.

Let $\theta(p, q)G$ denote the subgroup of G which consists of all elements of the form $\theta(p, q)g$, $g \in G$. In the definition of cochains given in § 2, we consider instead of $\Phi^m(X, G)$, the larger subgroup $\Phi_{(p,q)}^m(X, G)$ of $\Phi^m(X, G)$, which consists of all m -functions ϕ of X into G such that there exists an open set N_ϕ of X^{m+1} containing the diagonal ΔX^{m+1} with $\phi(x_0, \dots, x_m) \in \theta(p, q)G$ for every point $(x_0, \dots, x_m)$ of N_ϕ . Define the factor group

$$C_{(p,q)}^m(X, G) = \Phi^m(X, G) / \Phi_{(p,p)}^m(X, G)$$

to be the group of m -dimensional (p, q) -cochains of X over G . In terms of these cochains, the groups of §§ 6 and 8 can be defined without the assumption that $\theta(p, q)G = 0$.

Appendix B. A sketch of the singular theory.

Let X be a topological space and let $S(X)$ denote the singular complex of X as defined by EILENBERG, [1, p. 420]. Let $C_m(X)$ denote the group of integral singular m -chains in X .

Let p be a given positive integer and consider the ordered geometric m -simplex ($m \geq p$)

$$s = \langle v_0, v_1, \dots, v_m \rangle.$$

From the $m + 1$ integers $0, 1, \dots, m$, let us take arbitrary p distinct integers $i_1, \dots, i_p$ satisfying

$$i_1 < i_2 < \dots < i_p.$$

Let the $m - p + 1$ remaining integers be denoted by $j_0, \dots, j_{m-p}$ with the order not disturbed. They determine an $(m - p)$ -dimensional face

$$s^{(i_1, \dots, i_p)} = \langle v_{j_0}, v_{j_1}, \dots, v_{j_{m-p}} \rangle$$

of s .

Given a singular m -simplex

$$T: s \rightarrow X,$$

consider the singular $(m - p)$ -simplices

$$T^{(i_1, \dots, i_p)}: s^{(i_1, \dots, i_p)} \rightarrow X$$

defined by the partial mappings $T^{(i_1, \dots, i_p)} = T|_{s^{(i_1, \dots, i_p)}}$. We define the p -th boundary of T to be

$$\partial^p T = \sum \text{sgn} \begin{pmatrix} 0, \dots, p-1, p, \dots, m \\ i_1, \dots, i_p, j_0, \dots, j_{m-p} \end{pmatrix} T^{(i_1, \dots, i_p)}$$

The summation ranges over all the possible choices of the integers $i_1, \dots, i_p$ described above.

It is clear that $T_1 \equiv T_2$ implies that

$$T_1^{(i_1, \dots, i_p)} \equiv T_2^{(i_1, \dots, i_p)}$$

and therefore $\partial^p T_1 = \partial^p T_2$ in $C_{m-p}(X)$. Hence we get a homomorphism

$$\partial^p: C_m(X) \rightarrow C_{m-p}(X).$$

As in § 4, one might verify that

$$\partial^p \partial^q = \partial^q \partial^p = \theta(p, q) \partial^{p+q}$$

for any two positive integers p and q . Hence, one may define the *singular* (p, q) -homology groups and the *singular* (p, q) -cohomology groups.

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§ 1. *Introduction.*

We consider an infinite sequence of real numbers $x_1, x_2, \dots$. Then the infinite sequence of the rests modulo 1

$$x_1 - [x_1], x_2 - [x_2], \dots$$

(where $[x]$ denotes the largest integer $\leq x$) all belong to the unity-interval $\mathfrak{J}$ ($0 \leq x \leq 1$). The problem how these rests are distributed on $\mathfrak{J}$ in several cases led to the theory of uniform distribution. We mention the definition of a uniformly distributed sequence of numbers $x_1, x_2, \dots$:

If N is a positive integer, and γ is a fixed (independent of N) number satisfying $0 \leq \gamma \leq 1$, we denote by $N(\gamma)$ the number of those rests from the sequence $x_1, x_2, \dots, x_N$ which belong to the interval $0 \leq x < \gamma$. The sequence $x_1, x_2, \dots$ is then said to be uniformly distributed (mod 1) if

$$\lim_{N \rightarrow \infty} \frac{N(\gamma)}{N} = \gamma$$

holds for every γ satisfying $0 \leq \gamma \leq 1$.

If $f(t)$ is a real function, defined for $t \geq 0$, the mentioned theory may therefore be said to deal with the distribution of the values of $f(t)$ (mod 1) for t a positive integer. In the existing literature some attention has also been paid to the case that all the values of $f(t)$ ($t \geq 0$) are considered. To distinguish between the two cases, we shall speak in the first case of a **discrete** distribution (shortly denoted by *D-distribution*) and in the second case of a **continuous** distribution or *C-distribution* (modulo 1) of $f(t)$. As far as we know this problem has been considered only by H. WEYL [1], G. PÓLYA and G. SZEGÖ [2], and B. E. BUNGAARD [3]. WEYL proved the following theorem:

Let $\varphi_1(t), \varphi_2(t), \dots, \varphi_p(t)$ be a system of linearly independent polynomials. Then a point, moving in the p -dimensional Euclidean space R_p according to $x_i = \varphi_i(t)$ ($i = 1, 2, \dots, p$), where t is interpreted as the time, spends (in the limit) equal times in equal parts of R_p .

For the linear problem (all $\varphi_p(t)$ of the first degree) BUNGAARD proved an analogue (at the same time a generalisation) in the theory of groups. L. KUIPERS, in his dissertation [4], developed systematically the theory of *C-distribution*. Several extensions were given to the already existing definition of *C-distribution*. In the present and the next paper we give a summary of these new definitions, and we show how further extensions are possible.

§ 2. Definition of *CI*-uniform distribution.

We return for a moment to the case of *D*-distribution. Let the sequence $f(1), f(2), \dots$ be uniformly distributed (mod 1). This means, using a terminology which is clear, that the rests (mod 1) lying on an arbitrarily chosen interval $\mathfrak{B}$ of $\mathfrak{J}$ with length l form the l -th part of all rests. If we replace the interval $\mathfrak{B}$ by a finite number of intervals $\mathfrak{B}_1, \dots, \mathfrak{B}_k$ with lengths $l_1, \dots, l_k$, such that no two of these intervals overlap, then the rests (mod 1) belonging to $\mathfrak{B}_1 + \dots + \mathfrak{B}_k$, form the $(l_1 + \dots + l_k)$ -th part of all rests. It is however not allowed to replace the set $\mathfrak{B}_1 + \dots + \mathfrak{B}_k$ in this argument by an arbitrarily chosen measurable pointset $\mathfrak{E}$ ($\mathfrak{E} \subset \mathfrak{J}$) with measure $\|\mathfrak{E}\|$. For the set of all rests (mod 1) of our sequence is enumerable, and therefore has measure zero. Hence $\|\mathfrak{E}\|$ does not change if we add all the rests to $\mathfrak{E}$ or remove them from $\mathfrak{E}$. It follows that if the sequence is uniformly distributed (mod 1), we cannot conclude that for every measurable $\mathfrak{E}$ the rests (mod 1) belonging to $\mathfrak{E}$ form the $\|\mathfrak{E}\|$ -th part of all rests.

As in the case of *D*-distribution we restrict ourselves to the values of $f(t)$ for $t \geq 0$, where we now assume $f(t)$ to be measurable in the sense of Borel-Lebesgue. For the case of *C*-distribution it is possible to give an analogue of the definition of *D*-distribution in more than one way. In each of these definitions we introduce as an analogue of the expression $N(\gamma)$ (cf. § 1) a Lebesgue-integral denoting the measure of the set of points t ($0 \leq t \leq N$) for which $f(t) - [f(t)]$ lies in a measurable subset of $\mathfrak{J}$. Using this integral, KUIPERS gave the following definition of *CI*-uniform distribution (mod 1) of $f(t)$.

Definition I.

For any given pair of real numbers α, β satisfying

$$0 \leq \alpha \leq \beta \leq 1,$$

we denote by $\theta(\alpha, \beta; f(t))$ ($f(t)$ being a measurable function of t for $t \geq 0$) the following characteristic function:

$$\theta(\alpha, \beta; f(t)) = 1 \text{ if } \alpha \leq f(t) < \beta \pmod{1},$$

$$\theta(\alpha, \beta; f(t)) = 0 \text{ elsewhere.}$$

Then, if for any pair α, β the function $f(t)$ satisfies

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \theta(\alpha, \beta; f(t)) dt = \beta - \alpha, \quad . \quad . \quad . \quad . \quad (1)$$

$f(t)$ is said to be *C*¹-uniformly distributed (mod. 1).

Remarks. 1°. Let $\mathfrak{S}_T(\alpha, \beta)$ be the set of points t in

$$0 \leq t \leq T \text{ with } \alpha \leq f(t) < \beta \pmod{1}.$$

It is obvious that the integral in (1) represents $\|\mathfrak{S}_T(\alpha, \beta)\|$, hence (1) can be written as

$$\lim_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\alpha, \beta)\| = \beta - \alpha. \quad . \quad . \quad . \quad . \quad (2)$$

2°. That the measure $\|\mathfrak{S}_T(a, \beta)\|$ exists for every pair a, β with $0 \leq a \leq \beta \leq 1$ follows from the fact that for any non-negative integer n the set of points t in $0 \leq t \leq T$ for which

$$n + a \leq f(t) < n + \beta$$

is measurable. Hence $\mathfrak{S}_T(a, \beta)$ is measurable as a sum of measurable sets.

§ 3. Definition of *CII-uniform distribution*.

The definition of *CI-uniform distribution* can be extended if we do not consider one single interval $\mathfrak{B}$ of $\mathfrak{J}$, but a sum $\sum_{k=1}^{\infty} \mathfrak{B}_k$ of non-overlapping intervals, all contained in $\mathfrak{J}$.

Definition II.

Let $\mathfrak{S}_T(\mathfrak{B}_k)$ be the set of points t in $0 \leq t \leq T$ with $f(t) - [f(t)]$ in $\mathfrak{B}_k$, where again $f(t)$ is a measurable function for $t \geq 0$. Then $f(t)$ is called *CII-uniformly distributed (mod 1)* if, for any sum $\sum_{k=1}^{\infty} \mathfrak{B}_k$ of non-overlapping intervals, $f(t)$ satisfies

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left\| \sum_{k=1}^{\infty} \mathfrak{S}_T(\mathfrak{B}_k) \right\| = \left\| \sum_{k=1}^{\infty} \mathfrak{B}_k \right\|. \quad (3)$$

Remarks. 1°. Since every set $\mathfrak{S}_T(\mathfrak{B}_k)$ is measurable, the same is true of $\sum_{k=1}^{\infty} \mathfrak{S}_T(\mathfrak{B}_k)$.

2°. The intervals $\mathfrak{B}_k$ being non-overlapping, (3) can be written as

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=1}^{\infty} \left\| \mathfrak{S}_T(\mathfrak{B}_k) \right\| = \sum_{k=1}^{\infty} \left\| \mathfrak{B}_k \right\|. \quad (4)$$

3°. Obviously, if a measurable function $f(t)$ is *CII-uniformly distributed (mod 1)*, then it is also *CI-uniformly distributed*.

§ 4. Definition of *CIII-uniform distribution*.

A second generalisation of *CI-distribution* may be obtained by considering an arbitrary measurable set $\mathfrak{E}$ contained in $\mathfrak{J}$, instead of an interval $\mathfrak{B}$. This leads to

Definition III.

Let $\mathfrak{S}_T(\mathfrak{E})$ be the set of points t in $0 \leq t \leq T$ with $f(t) - [f(t)]$ in $\mathfrak{E}$, where again $f(t)$ is measurable for $t \geq 0$. Then $f(t)$ is called *CIII-uniformly distributed (mod 1)* if, for any measurable $\mathfrak{E}$, $f(t)$ satisfies

$$\lim_{T \rightarrow \infty} \frac{1}{T} * \left\| \mathfrak{S}_T(\mathfrak{E}) \right\| = \left\| \mathfrak{E} \right\|, \quad (5)$$

where $* \left\| \mathfrak{E} \right\|$ denotes the exterior measure of $\mathfrak{E}$.

Remarks. 1°. It follows immediately that if a function $f(t)$ is *CIII*-uniformly distributed (mod 1) $f(t)$ also satisfies the definitions I and II.

2°. If conversely a function satisfies Definition I, it is not necessary that it satisfies Definition III, as may be demonstrated by the following example:

Let $f(1), f(2), \dots$ be a *D*-uniformly distributed sequence (mod 1). Now we define $f(t)$ by

$$f(t) = f(n) \text{ with } n-1 < t \leq n \ (n=1, 2, \dots).$$

This function satisfies Definition I. For, if with T integer T^* denotes the number of (mod 1) reduced values of $f(1), f(2), \dots, f(T)$ which belong to an interval $\mathfrak{B}$ on $\mathfrak{J}$, we have

$$\frac{T^*}{T} = \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B})\|.$$

Hence

$$\lim_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B})\| = \lim_{T \rightarrow \infty} \frac{T^*}{T} = \|\mathfrak{B}\|.$$

In order to prove that $f(t)$ does not satisfy Definition III we observe that the set of the rests (mod 1) of all the values of $f(t)$ is an enumerable set (so with zero measure). Calling this set $\mathfrak{F}$, the relation (5) with $\mathfrak{E} = \mathfrak{F}$ does not hold, the left side being unity and the right side zero.

3°. If however a function $f(t)$ is *CII*-uniformly distributed, then $f(t)$ satisfies Definition III. We shall prove this statement in the next paragraph.

§ 5. Equivalence of *CII*- and *CIII*-uniform distribution.

We prove the following **theorem**:

If a measurable function $f(t)$ is CII-uniformly distributed, then $f(t)$ is also CIII-uniformly distributed.

The proof consists of two parts.

Part 1. If $f(t)$ is CII-uniformly distributed (mod 1) and if $\mathfrak{E}$ is an arbitrary measurable set on $\mathfrak{J}$ then we have

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{E})\| \leq \|\mathfrak{E}\|. \quad (6)$$

Proof. The set $\mathfrak{E}$ is contained in, and metrically equivalent with the limit of descending sequence of systems of non-overlapping intervals $\sum_{k=1}^{\infty} \mathfrak{B}_{n,k} \ (n=1, 2, \dots)$. For any positive ε we may find an $N = N(\varepsilon)$ such that for $n > N$

$$\mathfrak{E} \subset \sum_{k=1}^{\infty} \mathfrak{B}_{n,k} \text{ and } \left\| \sum_{k=1}^{\infty} \mathfrak{B}_{n,k} \right\| < \|\mathfrak{E}\| + \varepsilon. \quad (7)$$

Then

$$\mathfrak{S}_T(\mathfrak{E}) \subset \sum_{k=1}^{\infty} \mathfrak{S}_T(\mathfrak{B}_{n,k}) \text{ for every } T \geq 0, \text{ hence}$$

$$\frac{1}{T} \|\mathfrak{S}_T(\mathfrak{E})\| \leq \frac{1}{T} \left\| \sum_{k=1}^{\infty} \mathfrak{S}_T(\mathfrak{B}_{n,k}) \right\|. \quad (8)$$

From (7), (8) and our hypothesis follows now

$$\limsup_{T \rightarrow \infty} \frac{1}{T} * \|\mathfrak{S}_T(\mathfrak{G})\| \leq \|\mathfrak{G}\| + \varepsilon.$$

Therefore we have (6).

Part 2. Under the same hypothesis as in Part 1, we have

$$\liminf_{T \rightarrow \infty} \frac{1}{T} * \|\mathfrak{S}_T(\mathfrak{G})\| \geq \|\mathfrak{G}\|. \quad (9)$$

Proof. Considering the complementary set $\mathfrak{G}_1$ of $\mathfrak{G}$, thus the $\mathfrak{J}-\mathfrak{G}$, we have

$$* \|\mathfrak{S}_T(\mathfrak{G}_1)\| = T - \|\mathfrak{S}_T(\mathfrak{G})\|, \quad (10)$$

where $\|\mathfrak{J}\|$ is the interior measure of $\mathfrak{J}$. From Part 1 follows

$$\limsup_{T \rightarrow \infty} \frac{1}{T} * \|\mathfrak{S}_T(\mathfrak{G}_1)\| \leq \|\mathfrak{J}-\mathfrak{G}\| = 1 - \|\mathfrak{G}\|. \quad (11)$$

According to (10), the left side of (11) equals

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \{T - \|\mathfrak{S}_T(\mathfrak{G})\|\} = 1 - \liminf_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{G})\|,$$

so we have, using (11),

$$1 - \liminf_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{G})\| \leq 1 - \|\mathfrak{G}\|, \text{ or}$$

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{G})\| \geq \|\mathfrak{G}\|, \text{ and a fortiori}$$

$$\liminf_{T \rightarrow \infty} \frac{1}{T} * \|\mathfrak{S}_T(\mathfrak{G})\| \geq \|\mathfrak{G}\|.$$

The theorem follows now immediately from the results proved in Part 1 and Part 2.

§ 6. Tests.

In the present paper we study tests for the considered cases of distribution. For the D -uniform distribution we recall WEYL's tests [5]:

D-test a. *Necessary and sufficient for the D -uniform distribution (mod 1) of the sequence $f(1), f(2), \dots$ is that, for every function $w(t)$, Riemann-integrable on $\mathfrak{J}$, $f(x)$ satisfies the relation:*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^N w(f(x) - [f(x)]) = \int_0^1 w(t) dt.$$

More important is:

D-test b. *Necessary and sufficient for the D -uniform distribution (mod 1) of the sequence $f(1), f(2), \dots$ is that, for any integer $h \neq 0$, $f(x)$ satisfies*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x=1}^N e^{2\pi i h f(x)} = 0.$$

As the analogues of these tests for the case of C -uniform distribution we find

CI-test a. *Necessary and sufficient for the CI-uniform distribution (mod 1) of the measurable function $f(t)$ is that for every function $w(u)$, Riemann-integrable on $\mathfrak{J}$, $f(t)$ satisfies the relation*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T w(f(t) - [f(t)]) dt = \int_0^1 w(u) du.$$

CI-test b. *Necessary and sufficient for the CI-uniform distribution (mod 1) of the measurable function $f(t)$ is that, for any integer $h \neq 0$, $f(t)$ satisfies*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{2\pi i h f(t)} dt = 0.$$

It is obvious that a CII -test is also a $CIII$ -test, and conversely (cf. § 4, Remark 1, and § 5). KUIPERS [6] gave the following.

CIII-test a. *Necessary and sufficient for the CIII-uniform distribution (mod 1) of the measurable function $f(t)$ is that for every function $w(u)$, Lebesgue-integrable on $\mathfrak{J}$, $f(t)$ satisfies*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T w(f(t) - [f(t)]) dt = \int_0^1 w(u) du.$$

For practical purposes this test is not useful. It is our aim, therefore, to formulate a more suitable test for CII - (and thus for $CIII$ -) uniform distribution.

A $CIII$ -uniformly distributed function $f(t)$ is also CI -uniformly distributed. So, if $\mathfrak{B}$ is an arbitrary interval on $\mathfrak{J}$, we have:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B})\| = \|\mathfrak{B}\|, \quad (12)$$

according to (2). Let $\sum_{k=1}^{\infty} \mathfrak{B}_k$ be an arbitrary system of intervals on $\mathfrak{J}$, and let $f(t)$ be CII -uniformly distributed. Then, from (4),

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=1}^{\infty} \|\mathfrak{S}_T(\mathfrak{B}_k)\| = \sum_{k=1}^{\infty} \|\mathfrak{B}_k\|,$$

and, from (12),

$$\sum_{k=1}^{\infty} \|\mathfrak{B}_k\| = \sum_{k=1}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\|.$$

A necessary condition for $f(t)$ to be CII -uniformly distributed is therefore:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=1}^{\infty} \|\mathfrak{S}_T(\mathfrak{B}_k)\| = \sum_{k=1}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\|. \quad . . . (13)$$

It is easily seen that the conditions (12) and (13) together are also sufficient. Hence:

CIII-test b. *Necessary and sufficient for the CIII-uniform distribution*

(mod 1) of the measurable function $f(t)$ is that, for any integer $h \neq 0$, $f(t)$ satisfies the relation

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{2\pi i h f(t)} dt = 0, \quad (14)$$

and that for any system of intervals $\sum_{k=1}^{\infty} \mathfrak{B}_k$ on $\mathfrak{J}$, the relation (13) is satisfied.

It is obvious, that if $f(t)$ satisfies (14) a sufficient condition for $f(t)$ being *CII*- and *CIII*-uniformly distributed is the uniform convergence in T of the series

$$\sum_{k=1}^{\infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| \text{ for any } \sum_{k=1}^{\infty} \mathfrak{B}_k \text{ on } \mathfrak{J}.$$

In our next paper we shall give some applications of the use of this test for a special class of functions. Furthermore we shall give a generalisation of the definition of *CI*-uniform distribution (mod. 1).

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Bandung, 1 November 1949.

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Mathematics. — *Asymptotic C-distribution.* (Second communication.)
 By L. KUIPERS and B. MEULENBELD. (Communicated by Prof.
 J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 26, 1949.)

In the preceding paper (First communication [1]) we gave the definitions of C^I -, C^{II} - and C^{III} -uniform distribution (mod 1) of a measurable function $f(t)$ ($t \geq 0$) and deduced tests by which this C-distribution may be demonstrated.

In the present paper we first give some examples of the use of these tests. Furthermore we show how the definition of the C^I -uniform distribution (mod 1) can be generalized.

§ 1. *Application of the C^I -test on some special functions.*

Theorem 1. *The function $f(t) = \log t$ is not C^I -uniformly distributed (mod 1).*

Proof. For any integer $h \neq 0$ we have:

$$\int_0^T e^{2\pi i h \log t} dt = \int_{-\infty}^{\log T} e^{(2\pi i h + 1)u} du = \frac{T e^{2\pi i h \log T}}{2\pi i h + 1}.$$

Hence the expression

$$\frac{1}{T} \int_0^T e^{2\pi i h \log t} dt$$

will fail to have a limit for $T \rightarrow \infty$, which shows that the function $f(t) = \log t$ is not C^I -uniformly distributed (mod 1).

Theorem 2. *The function $f(t) = \sin t$ is not C^I -uniformly distributed (mod 1).*

Proof. Applying the test it is no restriction to limit ourselves to the case that T increases through positive integers, or more generally, through a sequence of positive numbers, increasing with a fixed difference. Putting $T = 2n\pi$ (n integer, ≥ 0), we have for any integer $h \neq 0$:

$$\frac{1}{T} \int_0^T e^{2\pi i h \sin t} dt = \sum_{k=0}^{n-1} \frac{1}{T} \int_{2k\pi}^{2(k+1)\pi} e^{2\pi i h \sin t} dt = \frac{1}{2\pi} \int_0^{2\pi} e^{2\pi i h \sin t} dt.$$

We shall prove now that for $h = 1$ the integral $I = \int_0^{2\pi} e^{2\pi i h \sin t} dt$ does not vanish. For the real part of the integrand we have:

$$\int_0^{2\pi} \cos(2\pi \sin t) dt = 2 \int_0^{\pi} \cos(2\pi \sin t) dt = 2\pi J_0(2\pi) \neq 0,$$

according to a well-known property of the Bessel-function $J_0(x)$. Hence $f(t) = \sin t$ is not C^I -uniform distributed (mod 1).

§ 2. *Application of the C^I -test on special classes of functions.*

The following Theorem 3 gives a whole class of functions which are not C^I -uniformly distributed (mod 1).

Theorem 3. *If $f(t)$ is a differentiable function, defined for $t \geq 0$, with $f(t) \rightarrow \infty$ for $t \rightarrow \infty$, and if $f'(t)$ is finite with $t f'(t) \rightarrow 0$ for $t \rightarrow \infty$, then $f(t)$ is not C^I -uniformly distributed (mod 1).*

Proof.

$$\frac{1}{T} \int_0^T e^{2\pi i h f(t)} dt = e^{2\pi i h f(T)} - \frac{2\pi i h}{T} \int_0^T t f'(t) e^{2\pi i h f(t)} dt. \quad (1)$$

From our assumptions it follows that, given a positive ε we may find a $T_0 = T_0(\varepsilon)$ such that

$$|t f'(t)| < \frac{1}{2} \varepsilon \quad \text{for } t \geq T_0.$$

Hence

$$\left| \frac{1}{T} \int_0^T t f'(t) e^{2\pi i h f(t)} dt \right| \leq \frac{1}{T} \left| \int_0^{T_0} t f'(t) e^{2\pi i h f(t)} dt \right| + \frac{1}{T} \left| \int_{T_0}^T t f'(t) e^{2\pi i h f(t)} dt \right|,$$

and, since $t f'(t)$ is bounded, the right hand side of this inequality is

$$< \frac{c T_0}{T} + \frac{1}{2} \varepsilon < \varepsilon \quad \text{for } T \geq T_1(\varepsilon) \geq T_0(\varepsilon).$$

If $T \rightarrow \infty$, the second term on the right side of (1) converges to zero, and the first term fails to have a limit.

Hence $f(t)$ is not C^I -uniformly distributed (mod 1).

As examples of the use of the C^I -test with positive results we may prove the following two theorems 4 and 5, which KUIPERS [2] proved in a more elementary way.

Theorem 4. *If $f(t)$ is a differentiable function, defined for $t \geq 0$, and if $f'(t) > 0$ and monotonically non-decreasing for $t \geq 0$, then $f(t)$ is C^I -uniformly distributed (mod 1).*

Proof.

$$\frac{1}{T} \int_0^T e^{2\pi i h f(t)} dt = \frac{1}{T} \int_{f(0)}^{f(T)} e^{2\pi i h u} F'(u) du,$$

where $t = F(u)$ is the inverse function of $u = f(t)$.

Using the second mean-value theorem, we have:

$$\frac{1}{T} \int_{f(0)}^{f(T)} F'(u) \cos 2\pi h u du = \frac{1}{T f'(T)} \int_{f(0)}^{f(\xi)} \cos 2\pi h u du$$

for a suitable chosen value of ξ in the interval $(0, T)$ and this expression converges to zero for $T \rightarrow \infty$. The same argument holds for the imaginary part of $\frac{1}{T} \int_0^T e^{2\pi i h f(t)} dt$, so the theorem is proved.

Theorem 5. If $f(t)$ is a differentiable function, defined for $t \geq 0$, and if $f'(t)$ is monotonically non-increasing for $t \geq 0$ with $t f'(t) \rightarrow \infty$ for $t \rightarrow \infty$, then $f(t)$ is C^I -uniformly distributed (mod 1).

Proof.

$$\frac{1}{T} \int_0^T e^{2\pi i h f(t)} dt = \frac{1}{T} \int_{f(0)}^{f(T)} e^{2\pi i h u} F'(u) du,$$

where $t = F(u)$ is the inverse function of $u = f(t)$.

Using the second mean-value theorem, the last expression equals

$$\frac{1}{T f'(T)} \int_{f(0)}^{f(\xi)} \cos 2\pi h u du, \quad \text{with } 0 < \xi < T,$$

and so tends to zero for $T \rightarrow \infty$, and similarly for the imaginary part. Hence $f(t)$ is C^I -uniformly distributed (mod 1).

§ 3. Applications of the C^{III} -test.

In the paper already mentioned [1], we introduced a function which is C^I -, but not C^{II} - and C^{III} -uniformly distributed (mod 1).

In the present paragraph we shall prove that the C^I -distributed functions of Theorems 4 and 5 are also C^{III} -distributed.

Theorem 6. Under the assumptions of Theorem 4 $f(t)$ is C^{III} -uniformly distributed (mod 1).

Proof. Let $\sum_{k=1}^{\infty} \mathfrak{B}_k$ be an arbitrary system of non-overlapping intervals $\mathfrak{B}_k$ on $\mathfrak{J}$. Then, with arbitrarily chosen $T \geq 0$, the integer N exists, such that $N \leq f(T) < N+1$ hence $F(N) \leq T < F(N+1)$, where $t = F(u)$ denotes the inverse function of $u = f(t)$. Let $\mathfrak{B}_k$ be the interval (α_k, β_k) , and $\mathfrak{S}_T(\mathfrak{B}_k)$ the set of points of $0 \leq t \leq T$ with $\alpha_k \leq f(t) < \beta_k \pmod{1}$. Then

$$\left. \begin{aligned} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| &\leq \frac{1}{T} \sum_{n=0}^N \{F(n+\beta_k) - F(n+\alpha_k)\} = \\ &= \frac{1}{T} \sum_{n=0}^N (\beta_k - \alpha_k) F'(n + \xi_{k,n}). \end{aligned} \right\} \quad (2)$$

for a suitable chosen value of $\xi_{k,n}$ in

$$n + \alpha_k < u < n + \beta_k \quad (n = 1, 2, \dots, N).$$

Since $f'(t)$ is monotonically non-decreasing, the last expression is

$$\begin{aligned} & \equiv \frac{(\beta_k - \alpha_k) \sum_{n=0}^N F'(n)}{F(N)} \equiv \frac{\beta_k - \alpha_k}{F(N)} \left\{ \int_0^N F'(u) du + F'(0) \right\} = \\ & = \frac{(\beta_k - \alpha_k) \{F(N) - F(0) + F'(0)\}}{F(N)} \equiv 2(\beta_k - \alpha_k), \end{aligned}$$

for T sufficiently large (independent of k).

Hence the series $\sum_{k=1}^{\infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\|$ is uniformly convergent in T , and by § 6 of [1] $f(t)$ is C^{III} -uniformly distributed (mod 1).

Theorem 7. *Under the assumptions of Theorem 5 $f(t)$ is C^{III} -uniformly distributed (mod 1).*

Proof. In this case the expression (2) is

$$\begin{aligned} & \equiv \frac{(\beta_k - \alpha_k) \sum_{n=1}^N F'(n)}{F(N)} + \frac{(\beta_k - \alpha_k) F(N+1)}{F(N)} \\ & \equiv (\beta_k - \alpha_k) \left\{ 1 + \frac{F(N+1)}{F(N)} \right\}. \end{aligned}$$

From our assumptions it is easily seen that $\frac{F(N+1)}{F(N)} \rightarrow 1$ for $T \rightarrow \infty$.

Hence

$$\frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| < 3(\beta_k - \alpha_k)$$

for $T > T_0$ (T_0 independent of k), which shows that the series

$$\sum_{k=1}^{\infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\|$$

is uniformly convergent in T for $T > T_0$, and so $f(t)$ is C^{III} -uniformly distributed (mod 1).

§ 4. Generalisation of the C^I -uniform distribution.

In the present paragraph we shall give an extension of the definition of C^I -uniform distribution. Whenever a function $f(t)$ is C^II -distributed, then, for any $\sum_{k=1}^{\infty} \mathfrak{B}_k$, $f(t)$ satisfies the relation:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=1}^{\infty} \|\mathfrak{S}_T(\mathfrak{B}_k)\| = \sum_{k=1}^{\infty} \|\mathfrak{B}_k\|. \quad \dots \dots \dots (3)$$

When however a function $f(t)$ is C^I - and not C^II -uniformly distributed, it is possible to indicate special sets $\sum_{k=1}^{\infty} \mathfrak{B}_k$ for which $f(t)$ satisfies (3).

In this case we call $f(t)$ C^{Ia} -distributed with respect to $\sum_{k=1}^{\infty} \mathfrak{B}_k$.

Definition. $f(t)$ is C^{Ia} -distributed (mod 1) with respect to a set $\sum_{k=1}^{\infty} \mathfrak{B}_k$, if $f(t)$ is C^I -uniformly distributed (mod 1), and if $f(t)$ satisfies:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=1}^{\infty} \|\mathfrak{S}_T(\mathfrak{B}_k)\| = \sum_{k=1}^{\infty} \|\mathfrak{B}_k\|.$$

It is obvious that for C^{Ia} -distribution the following test holds:

Cla-test. Necessary and sufficient for the C^{Ia} -distribution (mod 1) of $f(t)$ with respect to a set $\sum_{k=1}^{\infty} \mathfrak{B}_k$ is that $f(t)$ and $\sum_{k=1}^{\infty} \mathfrak{B}_k$ satisfy the relations:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{2\pi i h f(t)} dt = 0 \quad (h \text{ integer, } \neq 0)$$

and

$$\lim_{T \rightarrow \infty} \sum_{k=1}^{\infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| = \sum_{k=1}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\|.$$

Now we may prove the following

Theorem 8. Each C^I -uniformly distributed function $f(t)$ (mod 1) is C^{Ia} -distributed with respect to any $\sum_{k=1}^{\infty} \mathfrak{B}_k$ with one limitpoint *).

Proof. For any positive quantity ε there exists a positive integer n , such that

$$\sum_{k=n+1}^{\infty} \mathfrak{B}_k \subset \mathfrak{L}$$

where $\mathfrak{L}$ is a subinterval of $\mathfrak{J}$ with length ε . From

$$\sum_{k=1}^n \mathfrak{S}_T(\mathfrak{B}_k) \subset \sum_{k=1}^{\infty} \mathfrak{S}_T(\mathfrak{B}_k)$$

follows:

$$\sum_{k=1}^n \|\mathfrak{S}_T(\mathfrak{B}_k)\| \leq \sum_{k=1}^{\infty} \|\mathfrak{S}_T(\mathfrak{B}_k)\|;$$

hence

$$\begin{aligned} \limsup_{T \rightarrow \infty} \sum_{k=1}^{\infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| &= \lim_{T \rightarrow \infty} \sum_{k=1}^n \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| + \\ &+ \limsup_{T \rightarrow \infty} \sum_{k=n+1}^{\infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| \leq \sum_{k=1}^n \|\mathfrak{B}_k\| + \lim_{T \rightarrow \infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{L})\| < \\ &< \sum_{k=1}^{\infty} \|\mathfrak{B}_k\| + \varepsilon. \end{aligned}$$

*) That is, there exists a point x_0 on $\mathfrak{J}$ such that every interval, containing x_0 in its interior, also contains all but a finite number of the intervals $\mathfrak{B}_k$.

On the other hand

$$\begin{aligned} \liminf_{T \rightarrow \infty} \sum_{k=1}^{\infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| &\equiv \lim_{T \rightarrow \infty} \sum_{k=1}^n \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| = \sum_{k=1}^n \|\mathfrak{B}_k\| = \\ &= \sum_{k=1}^{\infty} \|\mathfrak{B}_k\| - \sum_{k=n+1}^{\infty} \|\mathfrak{B}_k\| \equiv \sum_{k=1}^{\infty} \|\mathfrak{B}_k\| - \varepsilon. \end{aligned}$$

So we have:

$$\lim_{T \rightarrow \infty} \sum_{k=1}^{\infty} \frac{1}{T} \|\mathfrak{S}_T(\mathfrak{B}_k)\| = \sum_{k=1}^{\infty} \|\mathfrak{B}_k\|.$$

and the theorem is proved.

Remark. It is evident that the same argument holds if we consider a set $\sum_{k=1}^{\infty} \mathfrak{B}_k$ with a finite number of limitpoints.

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Bandung, 16 November 1949.

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Mathematics. — *On power series with integral coefficients. II* ¹⁾. By C. G. LEKKERKERKER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 29, 1949.)

§ 4. In order to generalize the foregoing results I compute the following determinant ²⁾:

$$\begin{vmatrix} p_1 & (a+1)_1 p_1 \dots (a+1)_{r_1-1} p_1 \dots p_s & (a+1)_1 p_s \dots (a+1)_{r_s-1} p_s \\ p_1^2 & (a+2)_1 p_1^2 \dots (a+2)_{r_1-1} p_1^2 \dots p_s^2 & (a+2)_1 p_s^2 \dots (a+2)_{r_s-1} p_s^2 \\ \dots & \dots & \dots \\ p_1^r & (a+r)_1 p_1^r \dots (a+r)_{r_1-1} p_1^r \dots p_s^r & (a+r)_1 p_s^r \dots (a+r)_{r_s-1} p_s^r \end{vmatrix} \quad (6)$$

where s is a positive integer; $p_1, p_2, \dots, p_s$ are real or complex numbers; $r_1, r_2, \dots, r_s$ are positive integers; $r = \sum_{\sigma=1}^s r_\sigma$; and $(a+q)_t$ denotes the product $(a+q)(a+q-1) \dots (a+q-t+1)$. I shall denote the determinant (6) by $D(p_1, p_2, \dots, p_s)$ or by $D(p_1, p_2, \dots, p_s; r_1, r_2, \dots, r_s)$.

First I shall prove that the determinant (6) does not depend on the parameter a . Obviously $(a+q)_t$ can be developed as follows:

$$(a+q)_t = q^t + \alpha_{t1} q^{t-1} + \alpha_{t2} q^{t-2} + \dots + \alpha_{tt},$$

hence

$$(a+q)_t = q^t + \beta_{t1} (a+q)_{t-1} + \beta_{t2} (a+q)_{t-2} + \dots + \beta_{tt},$$

where the coefficients α and β do not depend on q , and therefore have the same value for those products $(a+q)_t$ which enter in a certain column of (6). I perform now on the first r_1 columns successively these operations:

$$\left. \begin{array}{l} \text{column}_{r_1} - \beta_{r_1-1,1} \text{column}_{r_1-1} - \beta_{r_1-1,2} \text{column}_{r_1-2} - \dots - \beta_{r_1-1,r_1-1} \text{column}_1 \\ \text{column}_{r_1-1} - \beta_{r_1-2,1} \text{column}_{r_1-2} - \beta_{r_1-2,2} \text{column}_{r_1-3} - \dots - \beta_{r_1-2,r_1-2} \text{column}_1 \\ \dots \\ \text{column}_3 - \beta_{21} \text{column}_2 - \beta_{22} \text{column}_1 \\ \text{column}_2 - \beta_{11} \text{column}_1 \end{array} \right\} \quad (7)$$

such that the q th element of the $(t+1)$ th column becomes $q^t p_1^q$

$$(t = 0, 1, \dots, r_1-1; q = 1, 2, \dots, r).$$

If I treat in the same manner every set of columns which contain a certain number p_σ , then the resulting determinant does not contain a .

¹⁾ Part I of this investigation was published in these Proceedings, 52, p. 740.

²⁾ A slight generalization of the determinant (6) with $s = 2$ or $s = 3$, due to BONOLIS, is mentioned by Sir THOMAS MUIR, The theory of Determinants in the historical order of development III, 460—461; see also: J. G. VAN DER CORPUT, Over eenige determinanten, Verhandelingen Kon. Akad. v. Wetensch., Amsterdam, XIV, No. 3 (1930).

Let $f_h(p) = p^{h-1}$ ($h = 1, 2, \dots, r$). Then the determinant (9) can be written

$$| \dots f_h(p_{\sigma 1}) \quad f_h(p_{\sigma 2}) \dots f_h(p_{\sigma r_\sigma}) \dots |.$$

Further I introduce the divided differences $[a_1, a_2, \dots, a_k] f_h$ ($k \geq 2$), defined by

$$[a_1, a_2] f_h = \frac{f_h(a_2) - f_h(a_1)}{a_2 - a_1},$$

$$[a_1, a_2, \dots, a_k] f_h = \frac{[a_1, a_2, \dots, a_{k-2}, a_k] f_h - [a_1, a_2, \dots, a_{k-2}, a_{k-1}] f_h}{a_k - a_{k-1}} \quad (k \geq 2).$$

Marking those values of σ , for which $r_\sigma > 1$, the following equality can be obtained by subtracting columns in the determinant (9):

$$\begin{aligned} | \dots f_h(p_{\sigma 1}) \quad [p_{\sigma 1}, p_{\sigma 2}] f_h \dots [p_{\sigma 1}, p_{\sigma r_\sigma}] f_h \dots | \\ = \frac{M(p_{11}, \dots, p_{sr_s})}{\prod_{\sigma=1}^s (p_{\sigma 2} - p_{\sigma 1}) (p_{\sigma 3} - p_{\sigma 1}) \dots (p_{\sigma r_\sigma} - p_{\sigma 1})}. \end{aligned}$$

Repeating the process it is found

$$\begin{aligned} | \dots f_h(p_{\sigma 1}) \quad [p_{\sigma 1}, p_{\sigma 2}] f_h \dots [p_{\sigma 1}, p_{\sigma 2}, \dots, p_{\sigma r_\sigma}] f_h \dots | \\ = \frac{M(p_{11}, \dots, p_{sr_s})}{\prod_{\sigma=1}^s M(p_{\sigma 1}, \dots, p_{\sigma r_\sigma})}. \end{aligned}$$

Now let

$$p_{\sigma \varrho} \rightarrow p_\sigma \quad (\varrho = 1, 2, \dots, r_\sigma; \sigma = 1, 2, \dots, s).$$

Then, by a well known property of divided differences,

$$[p_{\sigma 1}, p_{\sigma 2}, \dots, p_{\sigma \varrho}] f_h \rightarrow \frac{1}{(\varrho - 1)!} \frac{\partial^{\varrho-1}}{\partial p_\sigma^{\varrho-1}} f_h(p_\sigma) \quad (\varrho = 2, 3, \dots, r_\sigma; \sigma = 1, 2, \dots, s).$$

The left member of the last equality tends to

$$\frac{1}{\prod_{\varrho=1}^{r_1-1} \varrho! \prod_{\varrho=1}^{r_2-1} \varrho! \dots \prod_{\varrho=1}^{r_s-1} \varrho!} F(p_1, p_2, \dots, p_s)$$

and the right member to

$$\prod_{i>j} (p_i - p_j)^{r_i r_j}.$$

Hence it follows:

$$\begin{aligned} D(p_1, p_2, \dots, p_s; r_1, r_2, \dots, r_s) \\ = \prod_{\sigma=1}^s p_\sigma^{r_\sigma(r_\sigma+1)} \cdot \prod_{\sigma=1}^s \prod_{\varrho=1}^{r_\sigma-1} \varrho! \cdot \prod_{i>j} (p_i - p_j)^{r_i r_j}. \end{aligned} \quad (10)$$

§ 5. Let $p_1, p_2, \dots, p_s$ be different complex numbers with non-negative imaginary part and with $0 < |p_\sigma| < 1$ ($\sigma = 1, 2, \dots, s$). Let $r_1, r_2, \dots, r_s$ be s positive integers. Let $\sum_{\sigma=1}^s r_\sigma = r$. If $\eta_{\sigma\varrho}$ ($\varrho = 1, 2, \dots, r_\sigma$; $\sigma = 1, 2, \dots, s$) are r real or complex numbers, then there exists a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with bounded integral coefficients a_n , so that $f^{(\varrho-1)}(p_\sigma) = \eta_{\sigma\varrho}$ for every pair σ, ϱ .

Proof. If z is a complex number or vector, I denote the adjoint number resp. vector by $\bar{z}$. If a number p_σ is non-real, I add to $p_1, p_2, \dots, p_s$ the number $\bar{p}_\sigma$ and prescribe the values $f^{(\tau-1)}(\bar{p}_\sigma) = \bar{\eta}_{\sigma\varrho}$ ($\varrho = 1, 2, \dots, r_\sigma$). After this being performed for every σ when p_σ is non-real, I use for convenience the same notations and indices. Hence we may suppose that for every $\sigma = 1, 2, \dots, s$ there exists an integer τ with $1 \leq \tau \leq s$, so that

$$\bar{p}_\sigma = p_\tau, \quad r_\sigma = r_\tau, \quad \bar{\eta}_{\sigma\varrho} = \eta_{\tau\varrho} \quad (\varrho = 1, 2, \dots, r_\sigma).$$

I introduce the square matrices

$$II_m = \begin{bmatrix} p_1 & ((m-1)r+1)_1 & p_1 \dots ((m-1)r+1)_{r_1-1} & p_1 \dots p_s \dots ((m-1)r+1)_{r_s-1} & p_s \\ p_1^2 & ((m-1)r+2)_1 & p_1^2 \dots ((m-1)r+2)_{r_1-1} & p_1^2 \dots p_s^2 \dots ((m-1)r+2)_{r_s-1} & p_s^2 \\ p_1^3 & ((m-1)r+3)_1 & p_1^3 \dots ((m-1)r+3)_{r_1-1} & p_1^3 \dots p_s^3 \dots ((m-1)r+3)_{r_s-1} & p_s^3 \\ \dots & \dots & \dots & \dots & \dots \\ p_1^r & (mr)_1 & p_1^r \dots (mr)_{r_1-1} & p_1^r \dots p_s^r \dots (mr)_{r_s-1} & p_s^r \end{bmatrix}$$

($m = 1, 2, \dots$) and

$$P = \begin{pmatrix} p_1^r & \dots & p_1^r & \dots & 0 \\ & \ddots & & \ddots & \\ & & p_1^r & \dots & \\ & & & \ddots & \\ & & & & p_s^r & \dots & p_s^r \\ 0 & & & & & \ddots & \end{pmatrix};$$

P contains in the principal diagonal r_1 elements p_1^r , r_2 elements $p_2^r, \dots, r_s$ elements p_s^r ; all other elements are zero. Since $\det II_m$ is the determinant (6) treated in § 4 with $a = (m-1)r$, $\det II_m$ does not depend on m and, on account of the formula (10) and the conditions made for $p_1, p_2, \dots, p_s$, is different from zero. Clearly $\det P$ also is different from zero.

If $a = (a_1, a_2, \dots, a_r)$, then also

$$a II_m P^{m-1} \quad (m = 1, 2, \dots) \quad \dots \quad (11)$$

is a r -dimensional vector, whose components have the form

$$a_1 ((m-1)r+1)_{\varrho-1} p_\sigma^{(m-1)r+1} + a_2 ((m-1)r+2)_{\varrho-1} p_\sigma^{(m-1)r+2} + \dots + a_r (mr)_{\varrho-1} p_\sigma^{mr}, \quad \left. \vphantom{a_1} \right\} \quad (12)$$

where for the pair σ, ϱ must be taken successively

$$1, 1; 1, 2; \dots; 1, r_1; \dots; s, 1; \dots; s, r_s \quad (r = r_1 + r_2 + \dots + r_s)$$

and where the coefficients $((m-1)r+1)_{\varrho-1}, \dots, (mr)_{\varrho-1}$ for $\varrho = 1$ are taken equal to 1. If I denote for every $\sigma = 1, 2, \dots, s$ the component with indices $\sigma, 1$ by β_σ , then the components of (11) are successively

$$\beta_1, p_1 \frac{d\beta_1}{dp_1}, \dots, p_1^{r_1-1} \frac{d^{r_1-1}\beta_1}{dp_1^{r_1-1}}, \dots, \beta_s, \dots, p_s^{r_s-1} \frac{d^{r_s-1}\beta_s}{dp_s^{r_s-1}}. \quad (13)$$

The coefficients a_n of the series $\sum_{n=0}^{\infty} a_n x^n$ are determined in the following manner. Put $a_0 = 0$. In order to define the vector $a^{(1)}$, consisting of the integers $a_1, a_2, \dots, a_r$, I consider first the vector $\omega^{(0)}$ with the components

$$\eta_{11}, p_1 \eta_{12}, \dots, p_1^{r_1-1} \eta_{1r_1}, \dots, \eta_{s1}, \dots, p_s^{r_s-1} \eta_{sr_s}$$

and I write

$$\omega^{(0)} \Pi_1^{-1} = u^{(1)}.$$

I shall prove that this vector $u^{(1)}$ has real components $u_1, u_2, \dots, u_r$. We have

$$u^{(1)} \Pi_1 = \omega^{(0)}.$$

Writing out this equation in full it yields, if we recall the definition of the vector $\omega^{(0)}$ and use (11) and (12) with $m = 1$, $\alpha = u^{(1)}$:

$$u_1 \cdot (1)_{\varrho-1} p_\sigma + u_2 \cdot (2)_{\varrho-1} p_\sigma^2 + \dots + u_r \cdot (r)_{\varrho-1} p_\sigma^r = p_\sigma^{\varrho-1} \eta_{\sigma\varrho} \\ (\varrho = 1, 2, \dots, r_s; \sigma = 1, 2, \dots, s).$$

Hence it follows

$$\bar{u}_1 \cdot (1)_{\varrho-1} \bar{p}_\sigma + \bar{u}_2 \cdot (2)_{\varrho-1} \bar{p}_\sigma^2 + \dots + \bar{u}_r \cdot (r)_{\varrho-1} \bar{p}_\sigma^r = \bar{p}_\sigma^{\varrho-1} \bar{\eta}_{\sigma\varrho}.$$

Now $\bar{p}_\sigma$ again is one of the numbers $p_1, p_2, \dots, p_s$, say p_τ , and if σ takes all values $1, 2, \dots, s$, then τ runs through a permutation of $1, 2, \dots, s$; moreover $\bar{\eta}_{\sigma\varrho} = \eta_{\tau\varrho}$. It follows

$$\bar{u}_1 \cdot (1)_{\varrho-1} p_\tau + \bar{u}_2 \cdot (2)_{\varrho-1} p_\tau^2 + \dots + \bar{u}_r \cdot (r)_{\varrho-1} p_\tau^r = p_\tau^{\varrho-1} \eta_{\tau\varrho} \\ (\varrho = 1, 2, \dots, r_s; \tau = 1, 2, \dots, s),$$

or

$$\overline{u^{(1)}} \Pi_1 = \omega^{(0)}, \quad \overline{u^{(1)}} = \omega^{(0)} \Pi_1^{-1},$$

hence $\overline{u^{(1)}} = u^{(1)}$.

Now, $u^{(1)}$ being a real vector, we may write $u^{(1)} = a^{(1)} + \omega^{(1)}$, where $a^{(1)}$ has integral components and $\omega^{(1)}$ is a real vector with $|\omega^{(1)}| \leq \frac{1}{2}$. We have

$$\omega^{(0)} = a^{(1)} \Pi_1 + \omega^{(1)} \Pi_1.$$

Now I define the vectors $a^{(1)}, a^{(2)}, \dots$ by induction. Suppose that the r -dimensional vectors $a^{(1)} = (a_1, a_2, \dots, a_r)$, $a^{(2)} = (a_{r+1}, a_{r+2}, \dots, a_{2r}), \dots$, $a^{(m)} = (a_{(m-1)r+1}, \dots, a_{mr})$; $\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(m)}$ are determined, so that

Since $\det (\Pi_{m+1}P)$ does not vanish the system (15) possesses a unique solution which can be written in the form

$$d_{\varrho\tau} = \frac{\Delta_{\varrho\tau}}{\det \Pi_{m+1} \cdot \det P} \quad (\varrho = 1, 2, \dots, r; \tau = 1, 2, \dots, r),$$

where $\Delta_{\varrho\tau}$ is given by the determinant

$$\begin{vmatrix} p_1^{r+1} & \dots p_1^{r+\tau-1} & p_1^{\varrho} & p_1^{r+\tau+1} & \dots p_1^{2r} \\ (mr+1)_1 p_1^{r+1} & \dots (mr+\tau-1)_1 p_1^{r+\tau-1} & ((m-1)r+\varrho)_1 p_1^{\varrho} & (mr+\tau+1)_1 p_1^{r+\tau+1} & \dots ((m+1)r)_1 p_1^{2r} \\ \dots & \dots & \dots & \dots & \dots \\ (mr+1)_{r-1} p_1^{r+1} & \dots (mr+\tau-1)_{r-1} p_1^{r+\tau-1} & ((m-1)r+\varrho)_{r-1} p_1^{\varrho} & (mr+\tau+1)_{r-1} p_1^{r+\tau+1} & \dots ((m+1)r)_{r-1} p_1^{2r} \\ \dots & \dots & \dots & \dots & \dots \\ p_s^{r+1} & \dots p_s^{r+\tau-1} & p_s^{\varrho} & p_s^{r+\tau+1} & \dots p_s^{2r} \\ \dots & \dots & \dots & \dots & \dots \\ (mr+1)_{r-1} p_s^{r+1} & \dots (mr+\tau-1)_{r-1} p_s^{r+\tau-1} & ((m-1)r+\varrho)_{r-1} p_s^{\varrho} & (mr+\tau+1)_{r-1} p_s^{r+\tau+1} & \dots ((m+1)r)_{r-1} p_s^{2r} \end{vmatrix}$$

If we reflect this determinant in the main diagonal, it proves to be equal to the determinant (8) (see § 4, remark), with

$$k_1 = r+1, k_2 = r+2, \dots, k_{\tau-1} = r+\tau-1, k_{\tau} = \varrho, k_{\tau+1} = r+\tau+1, \dots, k_r = 2r \\ a_1 = 1, a_2 = 2, \dots, a_{\tau-1} = \tau-1, a_{\tau} = -r+\varrho, a_{\tau+1} = \tau+1, \dots, a_r = r, a = mr.$$

Since a is the only number which involves m , $\Delta_{\varrho\tau}$ does not depend on m . Nor do $\det \Pi_{m+1}$ and $\det P$ depend on m . Hence the numbers $d_{\varrho\tau}$ do not depend on m .

Since $\omega^{(m)}$ is a r -dimensional vector $(\omega_1^{(m)}, \omega_2^{(m)}, \dots, \omega_r^{(m)})$, the vector $\omega^{(m)} \Pi_m P^{m-1}$, say $\eta^{(m)}$, has the form (11). If we denote its components by $\eta_{\sigma\varrho}^{(m)}$ ($\varrho = 1, 2, \dots, r$; $\sigma = 1, 2, \dots, s$), then by (12)

$$\eta_{\sigma\varrho}^{(m)} = \omega_1^{(m)} \cdot ((m-1)r+1)_{\varrho-1} p_{\sigma}^{(m-1)r+1} + \omega_2^{(m)} \cdot ((m-1)r+2)_{\varrho-1} p_{\sigma}^{(m-1)r+2} + \dots \\ \dots + \omega_r^{(m)} \cdot (mr)_{\varrho-1} p_{\sigma}^{mr}.$$

Hence, on account of

$$|\omega^{(m)}| \leq \frac{1}{2} \quad \text{and} \quad ((m-1)r+\lambda)_{\varrho-1} \leq (mr)^r \quad (\lambda = 1, 2, \dots, r),$$

we find

$$|\eta_{\sigma\varrho}^{(m)}| \leq \frac{1}{2} \cdot (mr)^r \sum_{\lambda=1}^r |p_{\sigma}|^{(m-1)r+\lambda};$$

because $|p_{\sigma}| < 1$ ($\sigma = 1, 2, \dots, s$) it follows that the components of $\eta^{(m)}$ tend to zero for $m \rightarrow \infty$.

The components of $a^{(m)} = (a_{(m-1)r+1}, \dots, a_{mr})$, are integers; on account of (11), (12), (13) the vector $a^{(m)} \Pi_m P^{m-1}$ has the components

$$\left. \begin{aligned} & a_{(m-1)r+1} p_{\sigma}^{(m-1)r+1} + \dots + a_{mr} p_{\sigma}^{mr} \\ & p_{\sigma} \frac{d}{dp_{\sigma}} \{ a_{(m-1)r+1} p_{\sigma}^{(m-1)r+1} + \dots + a_{mr} p_{\sigma}^{mr} \} \\ & \dots \\ & p_{\sigma}^{r_{\sigma}-1} \frac{d^{r_{\sigma}-1}}{dp_{\sigma}^{r_{\sigma}-1}} \{ a_{(m-1)r+1} p_{\sigma}^{(m-1)r+1} + \dots + a_{mr} p_{\sigma}^{mr} \} \end{aligned} \right\} \quad (\sigma = 1, 2, \dots, s).$$

Hence the components of $a^{(1)} \Pi_1 + a^{(2)} \Pi_2 P + \dots a^{(m)} \Pi_m P^{m-1}$ can be written

$$p_\sigma^{e-1} \frac{d^{e-1}}{dp_\sigma^{e-1}} \{a_1 p_\sigma + \dots + a_{mr} p_\sigma^{mr}\} \quad (\varrho = 1, 2, \dots, r; \sigma = 1, 2, \dots, s).$$

Recalling that $\omega^{(0)}$ was introduced as the vector with components $p_\sigma^{e-1} \eta_{\sigma\varrho}$ we conclude from (14) with $k = m$:

$$p_\sigma^{e-1} \eta_{\sigma\varrho} = p_\sigma^{e-1} \frac{d^{e-1}}{dp_\sigma^{e-1}} \{a_1 p_\sigma + \dots + a_{mr} p_\sigma^{mr}\} + \eta_{\sigma\varrho}^{(m)}.$$

Now $f(p_\sigma)$ is convergent and $|\eta^{(m)}| \rightarrow 0$ ($m \rightarrow \infty$). Hence

$$\eta_{\sigma\varrho} = \frac{d^{e-1}}{dp_\sigma^{e-1}} \sum_{n=0}^{\infty} a_n p_\sigma^n \quad (\varrho = 1, 2, \dots, r; \sigma = 1, 2, \dots, s).$$

§ 6. Let r be a positive integer. Let $A_1, A_2, A_3, \dots$ be a sequence of sets of real numbers and $g_1, g_2, g_3, \dots$ be a sequence of positive numbers such that for each $n = 1, 2, \dots$ each closed interval of length g_n contains at least one number of A_n . Let for every $m = 1, 2, \dots$ the maximum of the numbers

$$g_{(m-1)r+1}, g_{(m-1)r+2}, \dots, g_{mr}$$

be denoted by h_m . Let

$$\lim_{n \rightarrow \infty} \inf \sqrt[n]{\frac{1}{g_n}}$$

be positive or infinite.

If $p_1, p_2, \dots, p_r; \eta_1, \eta_2, \dots, \eta_r$ are real or complex numbers with the following properties:

- 1° $p_1, p_2, \dots, p_r$ are different,
- 2° for every $\varrho = 1, 2, \dots, r$ the complex adjoint number of p_ϱ again belongs to the set $p_1, p_2, \dots, p_r$ and if $\bar{p}_\varrho = p_\tau$, then also $\bar{\eta}_\varrho = \eta_\tau$,

$$3^\circ \quad 0 < |p_\varrho| < \lim_{n \rightarrow \infty} \inf \sqrt[n]{\frac{1}{g_n}} \quad (\varrho = 1, 2, \dots, r),$$

then there exists a power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$ and a positive number Q ,

which depends only on $p_1, p_2, \dots, p_r$, so that

- a) a_n belongs to A_n ($n = 1, 2, 3, \dots$),
- b) $|a_{mr+\varrho}| \leq Q h_m + \frac{1}{2} h_{m+1}$ ($\varrho = 1, 2, \dots, r; m = 1, 2, \dots$),
- c) $f(p_\varrho) = \eta_\varrho$ ($\varrho = 1, 2, \dots, r$).

Proof. I use the same matrices Π and P as in § 3. Again I consider the r -dimensional vector $\omega^{(0)} = (\eta_1, \eta_2, \dots, \eta_r)$. Because $\det \Pi \neq 0$ we can determine the vector $u^{(1)}$ from

$$\omega^{(0)} = u^{(1)} \Pi$$

or $u^{(1)} = \omega^{(0)} \Pi^{-1}$.

Since it can be proved in the same manner as in § 5 that $\overline{u^{(1)}}\Pi = \omega^{(0)}$ or $\overline{u^{(1)}} = \omega^{(0)}\Pi^{-1}$, $u^{(1)}$ is real and therefore can be written as $a^{(1)} + \omega^{(1)}$, where we can choose the components $a_1, a_2, \dots, a_r$, viz. $\omega_1^{(1)}, \omega_2^{(1)}, \dots, \omega_r^{(1)}$ of these last vectors in such a way, that a_1 belongs to A_1 , a_2 to A_2 , ..., a_r to A_r , and that

$$|\omega_1^{(1)}| \leq \frac{1}{2} g_1, \quad |\omega_2^{(1)}| \leq \frac{1}{2} g_2, \dots, |\omega_r^{(1)}| \leq \frac{1}{2} g_r.$$

It follows

$$\omega^{(0)} = a^{(1)}\Pi + \omega^{(1)}\Pi \quad \text{with} \quad |\omega^{(1)}| \leq \frac{1}{2} h_1.$$

Now I define the vectors $a^{(1)}, a^{(2)}, \dots$ by induction. Suppose that the r -dimensional vectors

$$a^{(1)} = (a_1, a_2, \dots, a_r), \quad a^{(2)} = (a_{r+1}, \dots, a_{2r}), \dots, a^{(m)} = (a_{(m-1)r+1}, \dots, a_{mr}) : \\ \omega^{(1)}, \omega^{(2)}, \dots, \omega^{(r)}$$

are determined, in such a way that

$$m \geq 1,$$

$$a_1 \text{ belongs to } A_1, a_2 \text{ to } A_2, \dots, a_{mr} \text{ to } A_{mr},$$

$$\omega^{(k)} \text{ is real with } |\omega^{(k)}| \leq \frac{1}{2} h_k \quad (k = 1, 2, \dots, m),$$

$$\omega^{(0)} = a^{(1)}\Pi + a^{(2)}\Pi P + \dots + a^{(k)}\Pi P^{k-1} + \omega^{(k)}\Pi P^{k-1} \quad (k = 1, 2, \dots, m). \quad (16)$$

Then it is easy to show that the vector $\omega^{(m)}\Pi P^{-1}\Pi^{-1}$, say $u^{(m+1)}$, is real and can be written as $a^{(m+1)} + \omega^{(m+1)}$, where $a^{(m+1)}$ is a vector with components $a_{mr+1}, \dots, a_{(m+1)r}$, successively belonging to $A_{mr+1}, \dots, A_{(m+1)r}$, and $\omega^{(m+1)}$ is a real r -dimensional vector with $|\omega^{(m+1)}| \leq \frac{1}{2} h_{m+1}$.

It follows

$$\omega^{(m)}\Pi P^{m-1} = a^{(m+1)}\Pi P^m + \omega^{(m+1)}\Pi P^m;$$

hence (16) is also true for $k = m + 1$.

Successively vectors $a^{(1)}, a^{(2)}, \dots$ with components $a_1, a_2, \dots, a_r, a_{r+1}, \dots, a_{2r}, \dots$ and vectors $\omega^{(1)}, \omega^{(2)}, \dots$ can be determined such that a_1 belongs to A_1 , a_2 to A_2 , ..., $|\omega^{(1)}| \leq \frac{1}{2} h_1$, $|\omega^{(2)}| \leq \frac{1}{2} h_2$, ...; and the relation (16) is true for every positive integer k .

In order to prove that the power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$ has the required properties b) and c) I denote the elements of the matrix $\Pi P^{-1}\Pi^{-1}$ by $d_{e\tau}$ ($e, \tau = 1, 2, \dots, r$) and I take

$$Q = \frac{r}{2} \operatorname{Max}_{e, \tau=1, 2, \dots, r} |d_{e\tau}|,$$

so that Q is positive and only depends on the numbers $p_1, p_2, \dots, p_r$. Then from

$$\omega^{(m)}\Pi P^{-1}\Pi^{-1} = u^{(m+1)} \quad \text{and} \quad |\omega^{(m)}| \leq \frac{1}{2} h_m$$

it clearly follows

$$|u^{(m+1)}| \leq \frac{1}{2} h_m \cdot r \operatorname{Max} |d_{e\tau}| = Q h_m,$$

hence, on account of $u^{(m+1)} = a^{(m+1)} + \omega^{(m+1)}$,

$$|a^{(m+1)}| \leq Q h_m + \frac{1}{2} h_{m+1} \quad (m = 1, 2, \dots).$$

The last inequality establishes the property *b*).

Let the vector $\omega^{(m)}$ consist of the components $\omega_1^{(m)}, \omega_2^{(m)}, \dots, \omega_r^{(m)}$. Then the components of the vector $\omega^{(m)} \Pi P^{m-1}$, say $\eta_\varrho^{(m)}$, are

$$\eta_\varrho^{(m)} = \omega_1^{(m)} p_\varrho^{(m-1)r+1} + \omega_2^{(m)} p_\varrho^{(m-1)r+2} + \dots + \omega_r^{(m)} p_\varrho^{mr} \quad (\varrho = 1, 2, \dots, r).$$

We have

$$|\omega_\tau^{(m)} p_\varrho^{(m-1)r+\tau}| \leq \frac{1}{2} h_m |p_\varrho|^{(m-1)r+\tau} \quad (\varrho, \tau = 1, 2, \dots, r).$$

Because $|p_\varrho| < \liminf_{n \rightarrow \infty} \sqrt[n]{\frac{1}{g_n}}$, we have

$$\limsup_{n \rightarrow \infty} \sqrt[n]{g_n |p_\varrho|^n} < 1 \quad \text{for every } \varrho = 1, 2, \dots, r.$$

It follows

$$g_n p_\varrho^n \rightarrow 0 \quad \text{for } n \rightarrow \infty;$$

hence

$$h_m p_\varrho^{(m-1)r+\tau} \rightarrow 0 \quad \text{for } m \rightarrow \infty \quad (\varrho, \tau = 1, 2, \dots, r).$$

We conclude, that $\eta_\varrho^{(m)}$ ($\varrho = 1, 2, \dots, r$) tends to zero for $m \rightarrow \infty$.

Finally we have, on account of the relation (16),

$$\eta_\varrho = a_1 p_\varrho + p_\varrho^2 + \dots + a_{mr} p_\varrho^{mr} + \eta_\varrho^{(m)}.$$

From property *b*) one can deduce that $x = p_\varrho$ belongs to the circle of convergence of the power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Hence $f(p_\varrho) = \eta_\varrho$ ($\varrho = 1, 2, \dots, r$), i.e. the property *c*) is valid.

The problem treated in § 3 is a particular case of the problem solved here. If we take in the theorem just proved p_ϱ and η_ϱ real ($\varrho = 1, 2, \dots, r$) and if every set A_n is identified with the set of all integers, then we obtain the result of § 3 once more.

The problem considered in this paragraph can be generalized so as to include also the case of § 5. The corresponding treatment does not require essentially new considerations and is omitted here.

Finally I give in the following example another application of the theorem of this paragraph.

Let A_n ($n = 1, 2, \dots$) consist of all integral multiples of $\frac{1}{n!}$. Then we can take $g_n = \frac{1}{n!}$, so that

$$\liminf_{n \rightarrow \infty} \sqrt[n]{\frac{1}{g_n}} = \infty \quad \text{and} \quad h_m = \frac{1}{\{(m-1)r+1\}!} \quad (m = 1, 2, \dots).$$

Applying our theorem we obtain easily the following result:

Let $p_1, p_2, \dots, p_r; \eta_1, \eta_2, \dots, \eta_r$ be real or complex numbers so that

- 1° every number of the set $p_1, p_2, \dots, p_r$ is different from zero,
- 2° $p_1, p_2, \dots, p_r$ are different,
- 3° every number $\bar{p}_\varrho$ ($\varrho = 1, 2, \dots, r$) belongs to the set $p_1, p_2, \dots, p_r$ and if $\bar{p}_\varrho = p_\tau$, then also $\bar{\eta}_\varrho = \eta_\tau$.

Then there exists a positive number Q_1 , and a power series

$$f(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n$$

with integral coefficients b_n , such that

$$|b_n| < Q_1 n^{2r-1} \quad (n = 1, 2, \dots)$$

and

$$f(p_\varrho) = \eta_\varrho \quad (n = 1, 2, \dots, r).$$

In connection with this example I remark, that it is not difficult to prove, that if p is a given real number, all numbers ω , which can be represented by a power series $\sum_{n=1}^{\infty} \frac{b_n}{n!} p^n$ with bounded, integral coefficients b_n , form a set of measure zero.

Mathematics. — *On Dyson's improvement of the Thue-Siegel theorem.*
By K. MAHLER. (Mathematics Department, University of Manchester.) (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 29, 1949.)

Two years ago, F. J. DYSON proved the following result ¹⁾:

"If ξ is an algebraic number of degree $n \geq 2$, if μ is a positive number, and if there are infinitely many rational numbers $\frac{p}{q}$ such that

$$p, q \text{ are integers, } q \geq 1, \left| \xi - \frac{p}{q} \right| < q^{-\mu},$$

then

$$\mu \leq \sqrt{2n}."$$

This result is stronger than that of C. L. SIEGEL ²⁾, namely

$$\mu \leq \min_{s=1,2,\dots,n-1} \left(\frac{n}{s+1} + s \right) < 2\sqrt{n}.$$

DYSON obtained his improved inequality by means of a new method for studying the zero points of a polynomial in two variables. As his own proof is given in a somewhat involved form, I present in this paper a simplified proof for his main lemma (Theorem 1). Moreover, since this proof is purely algebraic, I deal always with the case of an arbitrary constant field of characteristic zero. This restriction is a natural one, since neither Theorem 1, nor the Thue-Siegel theorem, hold generally for fields of positive characteristic.

P.S. Since the time earlier this year when I wrote the present paper, a new proof of DYSON's result has been published by TH. SCHNEIDER ³⁾. This proof applies the deeper arithmetical properties of divisibility and may prove more powerful ⁴⁾.

[1] In this paper, K denotes a fixed field of characteristic zero; $K[x]$, $K[y]$, and $K[x, y]$, are the rings of all polynomials in x , in y , or in x and y , respectively, with coefficients in K ; and $K(x)$ denotes the field of all rational functions in x with coefficients in K . The terms "dependent" and "independent" always mean, "linearly dependent" and "linearly independent" over K .

¹⁾ Acta Mathematica **79**, 225—240 (1947).

²⁾ Mathematische Zeitschrift **10**, 173—213 (1921).

³⁾ Mathematische Nachrichten **2**, 288—295 (1949).

⁴⁾ Still another proof and a generalization of Dyson's theorem was given by A. O. GELFOND (Vestnik MGU **9**, 3 (1948)), but I have not seen his paper.

[2] We define differentiation in $K(x)$ in the usual formal way. Let $u_0(x), u_1(x), \dots, u_{l-1}(x)$ be a finite set of elements of $K(x)$; the determinant

$$\left| \frac{d^\mu u_\lambda(x)}{dx^\mu} \right|_{\lambda, \mu=0, 1, \dots, l-1}$$

is then called the Wronski determinant of these elements and is denoted by

$$\langle u_0, u_1, \dots, u_{l-1} \rangle.$$

One easily verifies that if $\varphi(x)$ is any further element of $K(x)$, then

$$\langle \varphi u_0, \varphi u_1, \dots, \varphi u_{l-1} \rangle = \varphi(x)^l \langle u_0, u_1, \dots, u_{l-1} \rangle.$$

Lemma 1: *The Wronski determinant of any finite number of elements of $K(x)$ vanishes identically in x if, and only if, these elements are dependent.*

Proof: If

$$\sum_{\lambda=0}^{l-1} c_\lambda u_\lambda(x) \equiv 0, \quad \text{where } c_\lambda \in K,$$

then

$$\sum_{\lambda=0}^{l-1} c_\lambda \frac{d^\mu u_\lambda(x)}{dx^\mu} \equiv 0 \quad (\mu = 0, 1, \dots, l-1),$$

whence $\langle u_0, u_1, \dots, u_{l-1} \rangle \equiv 0$.

Next assume that $\langle u_0, u_1, \dots, u_{l-1} \rangle \equiv 0$; we must show that $u_0(x), u_1(x), \dots, u_{l-1}(x)$ are dependent. This assertion is obvious for $l=1$; assume it has already been proved for all systems of less than l rational functions. We may exclude the case that $u_0(x) \equiv 0$ since then the Wronski determinant certainly vanishes. Hence

$$\begin{aligned} u_0(x)^{-l} \langle u_0, u_1, \dots, u_{l-1} \rangle &= \left\langle 1, \frac{u_1}{u_0}, \frac{u_2}{u_0}, \dots, \frac{u_{l-1}}{u_0} \right\rangle = \\ &= \left\langle \frac{d(u_1/u_0)}{dx}, \frac{d(u_2/u_0)}{dx}, \dots, \frac{d(u_{l-1}/u_0)}{dx} \right\rangle \equiv 0. \end{aligned}$$

Therefore, by the induction hypothesis, there exist $l-1$ elements $c_1, c_2, \dots, c_{l-1}$ of K not all zero such that

$$c_1 \frac{d(u_1/u_0)}{dx} + c_2 \frac{d(u_2/u_0)}{dx} + \dots + c_{l-1} \frac{d(u_{l-1}/u_0)}{dx} \equiv 0.$$

Since the characteristic of K is zero, this implies that

$$c_0 + c_1 \frac{u_1(x)}{u_0(x)} + c_2 \frac{u_2(x)}{u_0(x)} + \dots + c_{l-1} \frac{u_{l-1}(x)}{u_0(x)} \equiv 0$$

for some element c_0 of K , whence the assertion.

[3] Let now $u_0(x), u_1(x), \dots, u_{l-1}(x)$ be a finite set of independent polynomials in $K[x]$, and assume that $u_0(x)$ is of the highest degree amongst these, the degree d_0 , say. Then constants $c_1, c_2, \dots, c_{l-1}$ in K can be found such that

$$u_\lambda^{(1)}(x) = c_\lambda u_0(x) + u_\lambda(x) \quad (\lambda = 1, 2, \dots, l-1)$$

are all of degree less than d_0 . Assume that $u_1(x)$ is of highest degree, d_1 say, amongst these $l-1$ polynomials. Then constants $c_2^{(1)}, c_3^{(1)}, \dots, c_{l-1}^{(1)}$ in K can be found such that the $l-2$ polynomials

$$u_\lambda^{(2)}(x) = c_\lambda^{(1)} u_1^{(1)}(x) + u_\lambda^{(1)}(x) \quad (\lambda = 2, 3, \dots, l-1)$$

are all of degree less than d_1 . Assume that $u_2^{(2)}(x)$ is of highest degree, d_2 say, amongst these polynomials. Then constants $c_3^{(2)}, c_4^{(2)}, \dots, c_{l-1}^{(2)}$ can be found such that the $l-3$ polynomials

$$u_\lambda^{(3)}(x) = c_\lambda^{(2)} u_2^{(2)}(x) + u_\lambda^{(2)}(x) \quad (\lambda = 3, 4, \dots, l-1)$$

are all of degree less than d_2 . Continuing in this way, we obtain a set of l polynomials

$$u_0(x), u_1^{(1)}(x), u_2^{(2)}(x), \dots, u_{l-1}^{(l-1)}(x)$$

of degrees

$$d_0, d_1, d_2, \dots, d_{l-1}$$

respectively, where

$$d_0 > d_1 > d_2 > \dots > d_{l-1}.$$

By the construction, each polynomial $u_\lambda^{(l)}(x)$ differs from $u_\lambda(x)$ only by a linear expression in $u_0(x), u_1(x), \dots, u_{\lambda-1}(x)$ with coefficients in K . Hence, by a simple property of determinants, the identity

$$\langle u_0, u_1, \dots, u_{l-1} \rangle = \langle u_0, u_1^{(1)}, \dots, u_{l-1}^{(l-1)} \rangle$$

holds.

Lemma 2: Let $u_0(x), u_1(x), \dots, u_{l-1}(x)$ be polynomials in $K[x]$ of degrees not greater than d . Then the Wronski determinant

$$\langle u_0, u_1, \dots, u_{l-1} \rangle$$

is a polynomial of degree not greater than $l(d-l+1)$.

Proof: It suffices to prove the assertion when the polynomials are independent. The polynomials

$$u_0(x), u_1^{(1)}(x), \dots, u_{l-1}^{(l-1)}(x),$$

as just constructed, have degrees

$$d_0 \leq d-0, d_1 \leq d-1, \dots, d_{l-1} \leq d-(l-1).$$

Furthermore, the Wronski determinant $\langle u_0, u_1^{(1)}, \dots, u_{l-1}^{(l-1)} \rangle$ is a sum of $l!$ terms of the form

$$\mp \frac{d^{i_0} u_0(x)}{dx^{i_0}} \frac{d^{i_1} u_1^{(1)}(x)}{dx^{i_1}} \dots \frac{d^{i_{l-1}} u_{l-1}^{(l-1)}(x)}{dx^{i_{l-1}}}$$

where $i_0, i_1, \dots, i_{l-1}$ run over all permutations of $0, 1, \dots, l-1$. Each such term is of degree

$$\sum_{\lambda=0}^{l-1} (d_\lambda - i_\lambda) = \sum_{\lambda=0}^{l-1} \{d_\lambda - (l - \lambda - 1)\} \leq \sum_{\lambda=0}^{l-1} \{(d - \lambda) - (l - \lambda - 1)\} = l(d - l + 1),$$

whence the assertion.

[4] If $P(x, y)$ is any polynomial in $K[x, y]$, then write

$$P_{ij}(x, y) = \frac{\partial^{i+j} P(x, y)}{i! j! \partial x^i \partial y^j} \quad (i, j = 0, 1, 2, \dots).$$

We denote by r and s two positive integers which will be fixed in the next section, by ξ and η two elements of K , and by ϑ a non-negative real number. We then say that $P(x, y)$ is at least of index ϑ at (ξ, η) if

$$P_{ij}(\xi, \eta) = 0 \text{ for } i \geq 0, j \geq 0, \frac{i}{r} + \frac{j}{s} < \vartheta;$$

in the special case $\vartheta = 0$, there are no conditions.

This definition can be replaced by an equivalent one, as follows. Denote by z an indeterminate. Then

$$P(\xi + xz^s, \eta + yz^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P_{ij}(\xi, \eta) x^i y^j z^{rs(\frac{i}{r} + \frac{j}{s})} = P\langle z \rangle \text{ say,}$$

becomes a polynomial in z with coefficients in $K[x, y]$. This formula shows that $P(x, y)$ is at least of index ϑ at (ξ, η) if, and only if, $P\langle z \rangle$ is divisible by $z^{rs\vartheta}$ (i.e. all powers of z occurring in $P\langle z \rangle$ must have exponents not less than $rs\vartheta$). If we multiply several such expressions

$$P_0\langle z \rangle, P_1\langle z \rangle, \dots, P_{l-1}\langle z \rangle$$

which are divisible by

$$z^{rs\vartheta_0}, z^{rs\vartheta_1}, \dots, z^{rs\vartheta_{l-1}},$$

respectively, then the product is divisible by

$$z^{rs(\vartheta_0 + \vartheta_1 + \dots + \vartheta_{l-1})}.$$

Therefore the following result holds:

Lemma 3: If, for $\lambda = 0, 1, \dots, l-1$, the polynomial $P_\lambda(x, y)$ in $K[x, y]$ is at least of index ϑ_λ at (ξ, η) , then

$$P_0(x, y) P_1(x, y) \dots P_{l-1}(x, y)$$

is at least of index

$$\vartheta_0 + \vartheta_1 + \dots + \vartheta_{l-1}$$

at (ξ, η) .

[5] From now on,

$$R(x, y) = \sum_{h=0}^r \sum_{k=0}^s R_{hk} x^h y^k \not\equiv 0$$

is a fixed polynomial in $K[x, y]$ of degrees not greater than r in x and s in y ; here r and s are given positive integers. We further denote by

$$\theta_0, \theta_1, \dots, \theta_n \quad (n \geq 0)$$

a finite number of real numbers satisfying

$$0 < \theta_f \leq 1 \quad (f = 0, 1, \dots, n),$$

and by

$$\xi_0, \xi_1, \dots, \xi_n \quad \text{and} \quad \eta_0, \eta_1, \dots, \eta_n$$

two sets, each of $n+1$ elements of K , such that no two elements of the same set are equal.

Throughout this note, we make the assumption that $R(x, y)$ is, for $f = 0, 1, \dots, n$, at least of index θ_f at (ξ_f, η_f) , so that

$$R_{ij}(\xi_f, \eta_f) = 0 \quad \text{if} \quad i \geq 0, j \geq 0, \frac{i}{r} + \frac{j}{s} < \theta_f, f = 0, 1, \dots, n.$$

[6] Since

$$R(x, y) = \sum_{k=0}^s \left(\sum_{h=0}^r R_{hk} x^h \right) y^k,$$

the polynomial can be written in the form

$$R(x, y) = \sum_{\lambda=0}^{l-1} u_{\lambda}(x) v_{\lambda}(y),$$

where the u 's are elements of $K[x]$ of degrees not greater than r , the v 's are polynomials in $K[y]$ of degrees not greater than s , and where

$$1 \leq l \leq \min(r, s) + 1.$$

Amongst all representations of this form, select one for which the number l of terms is a minimum. Then both the l polynomials

$$u_0(x), u_1(x), \dots, u_{l-1}(x),$$

and the l polynomials

$$v_0(y), v_1(y), \dots, v_{l-1}(y),$$

are independent. For if, say, the u 's are not independent, then we may assume that $u_{l-1}(x)$ can be written as

$$u_{l-1}(x) = \sum_{\lambda=0}^{l-2} a_{\lambda} u_{\lambda}(x)$$

where the coefficients a_{λ} lie in K ; therefore

$$R(x, y) = \sum_{\lambda=0}^{l-2} u_{\lambda}(x) \{v_{\lambda}(y) + a_{\lambda} v_{l-1}(y)\}$$

becomes a sum of only $l-1$ terms, contrary to the definition of l .

We conclude therefore from Lemma 1 that neither of the two Wronski determinants

$$U(x) = \langle u_0(x), u_1(x), \dots, u_{l-1}(x) \rangle \text{ and } V(y) = \langle v_0(y), v_1(y), \dots, v_{l-1}(y) \rangle$$

vanishes identically. Moreover, by Lemma 2,

$$U(x) \text{ is at most of degree } l(r-l+1) \text{ in } x,$$

and

$$V(y) \text{ is at most of degree } l(s-l+1) \text{ in } y.$$

[7] Denote by $(x - \xi_f)^{r_f}$, where $f = 0, 1, \dots, n$, the highest power of $x - \xi_f$ dividing $U(x)$, and by $(y - \eta_f)^{s_f}$, where $f = 0, 1, \dots, n$, the highest power of $y - \eta_f$ dividing $V(y)$. Since all the ξ 's and also all the η 's are different, $U(x)$ is divisible by

$$\prod_{f=0}^n (x - \xi_f)^{r_f},$$

and $V(y)$ is divisible by

$$\prod_{f=0}^n (y - \eta_f)^{s_f}.$$

Therefore, on comparing the degrees, we obtain the two inequalities,

$$\left. \begin{aligned} r_0 + r_1 + \dots + r_n &\leq l(r-l+1), \\ s_0 + s_1 + \dots + s_n &\leq l(s-l+1). \end{aligned} \right\} \dots \dots \dots (I)$$

[8] We next introduce the determinant

$$W(x, y) = |R_{x\mu}(x, y)|_{x, \mu=0, 1, \dots, l-1}.$$

Since

$$R_{x\mu}(x, y) = \frac{1}{x! \mu!} \sum_{\lambda=0}^{l-1} u_{\lambda}^{(x)}(x) v_{\lambda}^{(\mu)}(y),$$

the product rule of determinants leads to the identity,

$$U(x) V(y) = \{1! 2! \dots (l-1)!\}^2 W(x, y),$$

so that also $W(x, y)$ does not vanish identically.

[9] Let f be one of the indices $0, 1, \dots, n$. Then, by hypothesis, $R(x, y)$ is at least of index θ_f at (ξ_f, η_f) ; therefore $R_{x\mu}(x, y)$ is at least of index

$$\max\left(0, \theta_f - \frac{x}{r} - \frac{\mu}{s}\right)$$

at (ξ_f, η_f) .

Now $W(x, y)$ is a sum of $l!$ terms of the form

$$\mp R_{i_0}(x, y) R_{i_1}(x, y) \dots R_{i_{l-1}l-1}(x, y),$$

where $i_0, i_1, \dots, i_{l-1}$ run over all permutations of $0, 1, \dots, l-1$. By Lemma 3, such a term is at least of index

$$\begin{aligned} \sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{i_\lambda}{r} - \frac{\lambda}{s}\right) &\geq \sum_{\lambda=0}^{l-1} \max\left(-\frac{i_\lambda}{r}, \theta_f - \frac{i_\lambda}{r} - \frac{\lambda}{s}\right) = \\ &= \sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) - \sum_{\lambda=0}^{l-1} \frac{i_\lambda}{r} \end{aligned}$$

at (ξ_f, η_f) . Since

$$\sum_{\lambda=0}^{l-1} \frac{i_\lambda}{r} = \sum_{\lambda=0}^{l-1} \frac{\lambda}{r} = \frac{l(l-1)}{2r},$$

the whole determinant $W(x, y)$ is therefore also at least of index

$$\sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) - \frac{l(l-1)}{2r}$$

at (ξ_f, η_f) .

[10] On the other hand, $U(x)V(y)$ is divisible exactly by

$$(x - \xi_f)^{r_f} (y - \eta_f)^{s_f},$$

so that

$$\frac{\partial^{i+j} \{U(x)V(y)\}}{i! j! \partial x^i \partial y^j} \bigg|_{x=\xi_f, y=\eta_f} \begin{cases} = 0 & \text{if } i \geq 0, j \geq 0, \frac{i}{r} + \frac{j}{s} < \frac{r_f}{r} + \frac{s_f}{s}, \\ \neq 0 & \text{if } i = r_f, j = s_f. \end{cases}$$

From the identity

$$U(x)V(y) = \{1! 2! \dots (l-1)!\}^2 W(x, y),$$

we therefore deduce the relations

$$\sum_{\lambda=0}^{l-1} \max\left(0, \theta_f - \frac{\lambda}{s}\right) - \frac{l(l-1)}{2r} \leq \frac{r_f}{r} + \frac{s_f}{s} \quad (f = 0, 1, \dots, n).$$

On adding these $n + 1$ inequalities and the two inequalities (I), we obtain the final inequality

$$\sum_{f=0}^n \sum_{\lambda=0}^{l-1} \max \left(0, \theta_f - \frac{\lambda}{s} \right) \leq (n+1) \frac{l(l-1)}{2r} + \frac{l(r-l+1)}{r} + \frac{l(s-l+1)}{s}, \quad (\text{II})$$

where now the unknown degrees r_f and s_f no longer occur.

[11] The double sum on the left-hand side of (II) is easily replaced by a simple one. Put

$$\Delta_f = \min ([\theta_f s] + 1, l) \quad (f = 0, 1, \dots, n),$$

so that

$$\max \left(0, \theta_f - \frac{\lambda}{s} \right) = \begin{cases} \theta_f - \frac{\lambda}{s} & \text{if } 0 \leq \lambda \leq \Delta_f - 1, \\ 0 & \text{if } \lambda \geq \Delta_f. \end{cases}$$

Therefore

$$\sum_{\lambda=0}^{l-1} \max \left(0, \theta_f - \frac{\lambda}{s} \right) = \sum_{\lambda=0}^{\Delta_f-1} \left(\theta_f - \frac{\lambda}{s} \right) = \frac{1}{s} \Delta_f \left(2\theta_f - \frac{\Delta_f - 1}{s} \right).$$

so that the left-hand side of (II) may be written as

$$\frac{1}{s} \sum_{f=0}^n \Delta_f \left(2\theta_f - \frac{\Delta_f - 1}{s} \right).$$

In order to simplify further, put

$$X = \frac{l}{s}, \quad X_f = \min (\theta_f, X) \quad (f = 0, 1, \dots, n).$$

Then

$$s X_f = \min (s \theta_f, s X) = \min (s \theta_f, l)$$

and

$$\Delta_f - 1 \leq s X_f \leq \Delta_f, \text{ hence } \Delta_f \left(2\theta_f - \frac{\Delta_f - 1}{s} \right) \geq s X_f (2\theta_f - X_f).$$

Therefore (II) implies that

$$\frac{s}{2} \sum_{f=0}^n X_f (2\theta_f - X_f) \leq (n+1) \frac{l(l-1)}{2r} + \frac{l(r-l+1)}{r} + \frac{l(s-l+1)}{s}.$$

Next, the right-hand side of this inequality may be written as

$$\begin{aligned} (n+1) \frac{l(l-1)}{2r} + \frac{l(r-l+1)}{r} + \frac{l(s-l+1)}{s} &= \left(2l - \frac{l^2}{s} \right) + \\ &+ \left(\frac{l}{s} + (n-1) \frac{l(l-1)}{2r} \right) = s(2X - X^2) \left\{ 1 + \frac{1}{2-X} \left(\frac{1}{s} + \frac{(n-1)(l-1)}{2r} \right) \right\}. \end{aligned}$$

Because, by [6],

$$l \leq \min(r, s) + 1 \leq s + 1,$$

the inequality becomes therefore

$$\sum_{f=0}^n X_f (2\theta_f - X_f) \leq 2 \{1 - (1 - X)^2\} \left\{ 1 + \frac{1}{2 - X} \left(\frac{1}{s} + \frac{(n-1)s}{2r} \right) \right\}.$$

[12] So far, r and s have been left arbitrary. Let now δ be a number satisfying

$$0 < \delta \leq 1,$$

and restrict r and s by the conditions,

$$s \geq \frac{5}{\delta} \geq 5, \quad r \geq \frac{5(n-1)s}{2\delta}.$$

Then

$$X = \frac{l}{s} \leq \frac{s+1}{s} \leq 1 + \frac{1}{s}, \quad 2 - X \geq \frac{1}{s}, \quad \frac{1}{s} \leq \frac{\delta}{5}, \quad \frac{(n-1)s}{2r} \leq \frac{\delta}{5},$$

so that

$$\frac{1}{2 - X} \left(\frac{1}{s} + \frac{(n-1)s}{2r} \right) \leq \frac{5}{4} \left(\frac{\delta}{5} + \frac{\delta}{5} \right) = \frac{\delta}{2},$$

and our inequality takes the simple form

$$\sum_{f=0}^n X_f (2\theta_f - X_f) \leq (2 + \delta) \{1 - (1 - X)^2\}.$$

But, for $f = 0, 1, \dots, n$,

$$X_f (2\theta_f - X_f) - \theta_f^2 \{1 - (1 - X)^2\} \equiv \theta_f^2 (1 - X)^2 - (\theta_f - X_f)^2$$

is not negative, since either $X \geq \theta_f$, when $X_f = \theta_f$ and

$$\theta_f^2 (1 - X)^2 - (\theta_f - X_f)^2 = \theta_f^2 (1 - X)^2 \geq 0;$$

or $X < \theta_f$, when $X_f = X$ and $X \leq 1$ and therefore

$$\theta_f^2 (1 - X)^2 - (\theta_f - X_f)^2 \equiv X(1 - \theta_f) \{ \theta_f(1 - X) + (\theta_f - X) \} \geq 0.$$

Hence

$$\{1 - (1 - X)^2\} \sum_{f=0}^n \theta_f^2 \leq \sum_{f=0}^n X_f (2\theta_f - X_f) \leq (2 + \delta) \{1 - (1 - X)^2\},$$

and since $(1 - X)^2 < 1$, we obtain finally the result,

$$\sum_{f=0}^n \theta_f^2 \leq 2 + \delta.$$

Our discussion has thus led us to the following theorem:

Theorem 1: Let $\delta, \theta_0, \theta_1, \dots, \theta_n$ be $n+2$ real numbers satisfying

$$0 < \delta \leq 1, \quad 0 < \theta_0 \leq 1, \quad 0 < \theta_1 \leq 1, \quad \dots, \quad 0 < \theta_n \leq 1,$$

and let r and s be two integers satisfying

$$s \geq \frac{5}{\delta}, \quad r \geq \frac{5(n+1)s}{2\delta}.$$

Let

$$R(x, y) \neq 0$$

be a polynomial of degrees not greater than r in x and s in y , with coefficients in a field K of characteristic zero; write

$$R_{ij}(x, y) = \frac{\partial^{i+j} R(x, y)}{i! j! \partial x^i \partial y^j} \quad (i, j = 0, 1, 2, \dots).$$

Further let

$$\xi_0, \xi_1, \dots, \xi_n \quad \text{and} \quad \eta_0, \eta_1, \dots, \eta_n$$

be two sets, each of $n+1$ elements of K , such that no two elements of the same set are equal. If now

$$R_{ij}(\xi_f, \eta_f) = 0 \quad \text{for } i \geq 0, \quad j \geq 0, \quad \frac{i}{r} + \frac{j}{s} < \theta_f, \quad f = 0, 1, \dots, n,$$

then

$$\theta_0^2 + \theta_1^2 + \dots + \theta_n^2 \leq 2 + \delta.$$

In a second paper, I shall prove an analogous theorem for polynomials of the form

$$\sum \sum R_{hk} x^h y^k \quad \left(h \geq 0, k \geq 0, \frac{h}{r} + \frac{k}{s} \leq 1 \right).$$

and apply this result to the study of the continued fractions of algebraic numbers.

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July 16, 1949.



Photo 1. Tomatoplant (Ailsa Craig) treated with T.I.B.A. by applying a lanolinpaste (50 mg T.I.B.A. + 5 ml water + 5 g woolfat) to the growing point. Development of a terminal inflorescence and an inflorescence in the axil of the first leaf. Photograph 42 days after the application.

Photo 2. Tomatoplant (Ailsa Craig) treated with T.I.B.A. by applying a lanolinpaste (50 mg T.I.B.A. + 5 ml water + 5 g woolfat) to the growing point. Inflorescences in the axils of the cotyledons. Photograph 31 days after the application.

Photo 3. Tip of tomatoplant (Ailsa Craig) treated with T.I.B.A. by applying a lanolinpaste (50 mg T.I.B.A. + 5 ml water + 5 g woolfat) to the cotyledons. After 8 hours the cotyledons were cut off. Photograph 59 days after the application. Terminal inflorescence developed. No sympodial bud.



Photo 4. Various structures of axillary shoots of T.I.B.A.-treated tomatoplants (Ailsa Craig). — a. Leafless shoot with fasciated tip. Standard application. Photograph 59 days after application. — b. Shoot in the axil of 4th leaf of a tomatoplant, treated with T.I.B.A. by applying a lanolin paste (50 mg T.I.B.A. + 5 ml water + 5 g woolfat) to the growing point. Photograph 41 days after the application. — c. Inflorescence in the axil of a cotyledon of a tomatoplant treated with T.I.B.A. by applying a lanolin paste (50 mg T.I.B.A. + 5 ml water + 5 g woolfat) to the growing point. Photograph 31 days after the application. — d. Axillary shoot of the 2nd leaf of a tomatoplant treated with T.I.B.A. by applying a lanolin paste (50 mg T.I.B.A. + 5 ml water + 5 g woolfat) to the growing point. Photograph 30 days after the application. — e. Axillary shoot of the 1st leaf of a tomatoplant treated with T.I.B.A. by applying a lanolin paste (50 mg T.I.B.A. + 5 ml water + 5 g woolfat) to the cotyledons for 48 hours.

Botany. — *The influence of 2, 3, 5 triiodobenzoic acid on the growing points of tomatoes.* By CHR^s. J. GORTER. (Publicatie No. 85 van het Laboratorium voor Tuinbouwplantenteelt te Wageningen.) (Communicated by Prof. W. H. ARISZ.)

(Communicated at the meeting of October 29, 1949.)

1. *Introduction.*

In the course of experiments concerning the correlation of the molecular structure and the activity of certain growth substances, ZIMMERMAN and HITCHCOCK (7) found a compound that caused abnormal flowering in tomatoplants.

This substance, 2, 3, 5 triiodobenzoic acid (in literature called: T.I.B.A. or T.I.B.) in low concentrations, varying from 20—500 mg/l, caused tomatoplants to produce inflorescences, located in anomalous positions and greater in number than in untreated plants. Authors call this chemical: a "formagen, affecting flowering, growth habits and correlation of organs".

All efforts to cause flowering by T.I.B.A. in plants, that were kept artificially vegetative by unfavourable photoperiods, failed [GALSTON (2), Soybeans, TUMANOV and LIZANDR (4) *Perilla*, *Pisum*, *Alfalfa*]. DE WAARD and Roodenburg (5) found "premature flowerbud initiation" in young tomato plants after the application of T.I.B.A.

In a recent paper ZIMMERMAN and HITCHCOCK (8) give more details about T.I.B.A. treated tomato plants, concluding that this chemical cannot act alone, as the effect varies with many circumstances, especially the growth of the plant. "It must depend upon materials made by the plant in order to cause vegetative tissue to produce flowers". They consider T.I.B.A. to have hormone characteristics.

In 1948 we started some experiments to see whether T.I.B.A. has florigenic activity. Our results agree with those of ZIMMERMAN and HITCHCOCK. On the ground of these experiments, we also have to come to the conclusion that the activity of T.I.B.A. in a plant is an indirect one. For once it interferes with the growth processes. Especially the interference with the growth relations in the growing points gives rise to abnormal structures and to the formation of inflorescences, abnormal in position and in number. In the present paper we deal with this abnormal structure of the growing points, in a second we hope to inform about some other phenomena caused by T.I.B.A., namely the initiation of fasciations, especially ringfasciations, the last phenomena occurring very rare in nature but frequently in T.I.B.A. treated plants.

2. *Experimental methods.*

The tomato varieties Ailsa Craig and Vetomold were used. In our

preliminary experiments T.I.B.A. was applied as a lanolin paste to the growing points or the leaves, as a spray on the leaves or in solution to the roots. Afterwards we came to the following standard application:

A lanolin paste of concentration: 50 mg T.I.B.A. + 5 ml water + 5 g woolfat, is applied to the cotyledons of the seedling. The plants are grown in 2-inch pots in a greenhouse at a temperature of $\pm 23^{\circ}\text{C}$.

The plants reacted in several different ways. Effects of T.I.B.A., which were always met with, were the following (see photo 1 and 2):

1. Reduction of the leafsurface and of development of leaflets.
2. Darker colour of the leaves.
3. Epinasty of leaves.
4. Reduction of internodal growth.
5. Inhibition of dominance of the tip.
6. Development of a terminal inflorescence (many flowers).
7. Development of axillary buds into abnormal structures.

(Even the axillary buds of the cotyledons can grow out into inflorescences).

Items 6 and 7 have to be discussed in more detail, as the question of the "florigenic" capacity of T.I.B.A. is dependent on these points.

3. Observations about the abnormal position of inflorescences.

a. Development of a terminal inflorescence.

The development of a terminal inflorescence in a T.I.B.A. treated tomato plant can be understood, when we take into consideration *the structure of the growing point of a normal plant*. Figures 1 a and b (half schematically) show growing points in two stages of development, in c we see the sequence

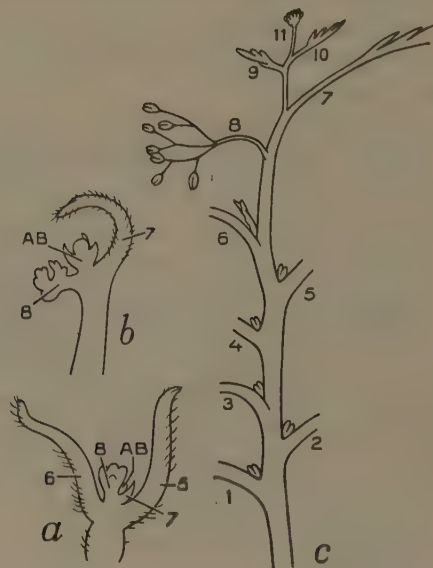


Fig. 1. Growingpoints of a normal tomatoplant in various stages of development.
Description in the text.

of leaves and inflorescences along the stem. The position of the leaves is alternate and the inflorescences appear after a number of leaves dependent on growth circumstances, 7 being the minimum. They are located on the internodes.

In the young growing point (a) there are two opposite leaves (5 and 6), above them a third leaf (7) and the inflorescence (8) terminally. The axillary bud (A.B.) in the axil of the last leaf is growing out, pushing the inflorescence aside (see b). Thus the stem is a true sympodium. The stem of the inflorescence and of the leaf beneath it in the growing point, grow together, the leaf stem growing faster. This difference in growth rates causes a reversal in the relative positions of leaf and inflorescence: the inflorescence can now be found below instead of above the leaf and on the internode. So in figure 1 c leaf 7 is the leaf that in the growing point can be found below the inflorescence. The leaves 9 and 10 and the inflorescence 11 are the organs of the sympodial bud.

In a T.I.B.A. treated tomato plant the terminal inflorescence bud remains terminal and grows out into a long stem with a terminal flower cluster. Moreover, the sympodial bud becomes an axillary shoot (or does not grow out at all) instead of developing as a continuation of the main stem (photo 3).

Hence, in connection with the structure of the growing point, the development of a terminal inflorescence in a T.I.B.A. treated plant, is not as abnormal as it at first appears. Only growth relations are altered, we have to do with a correlation phenomenon.

The development of the terminal inflorescence is not the result of the initiation of a new growing point.

b. Development of axillary buds into abnormal structures.

Depending on the method of application of T.I.B.A., on the concentration and on the growth and development of the plant, the axillary buds develop into one of the following structures (see photo 4):

- a. A long stem with a fasciated end (many flowers).
- b. A long stem with a flowerhead.
- c. A long stem with flowers.
- d. A long stem with connated flower clusters and leaves.
- e. A long stem with leaves at the end.

All transitional forms can be found.

It depends on several factors what develops; one of the most important is the growth rate of the plants. In general a vigorously growing plant frequently produces fasciations; plants growing in pots in the greenhouse make a large number of axillary flower stems a.s.o. ZIMMERMAN and HITCHCOCK (8) also state this correlation: "There was a direct relationship between vigour of the plants or rate of growth and the effect of the chemical".

It is apparent that T.I.B.A. can cause an axillary bud to develop into an inflorescence instead of into a leafy shoot.

4. *Observations about the number of flowers in T.I.B.A. treated plants.*

20 seedlings (Ailsa Craig) were treated according to the standard method by applying T.I.B.A. paste to the cotyledons. The seedlings had no flower primordia at the moment of the application.

Nine days after the application, the first abnormal axillary buds appeared; 55 days after the application the flowers were counted. The treated plants had 46 ± 6 flowers, the untreated ones 17 ± 3 flowers.

This proves that T.I.B.A. causes tomato plants to produce more flowers than untreated plants under the same conditions.

5. *The morphology of the growing points of T.I.B.A. treated plants.*

From the above mentioned facts it became probable that T.I.B.A. affected the growth relations in the meristems. We therefore investigated the morphology of the growing points.

The shape of the normal leaf forming growing apex (figure 2) is rather flat, part of a sphere, but not hemispheric. The leaves in their various stages of development can be found in succession.

The flower forming growing apex (figure 3) has a sort of peduncle and

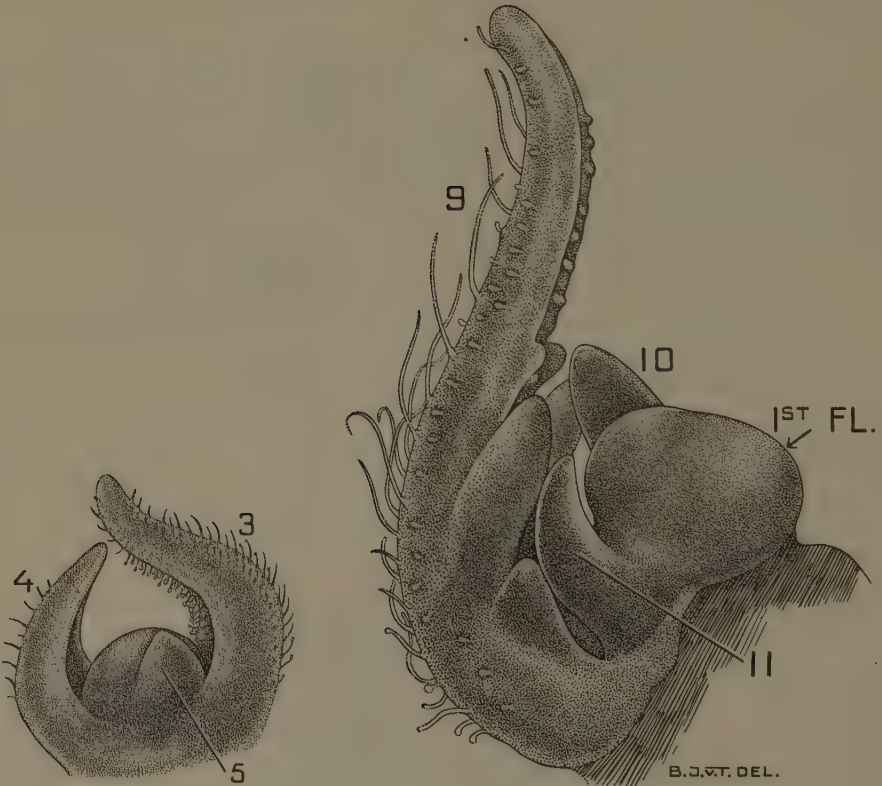


Fig. 2. Leafforming growing-point of a normal tomatoplant (magnification $80\times$).

Fig. 3. Flowerforming growing-point of a normal tomatoplant (magnification $80\times$).

a broadened head. The shape is quite different from that of the leaf forming growing apex. In an early stage of development an indication of dichotomy is found, one part gives rise to the top flower of the inflorescence, the other half dichotomizes again making the second flower a.s.o.

The shape of the *terminal growing apex* of a *T.I.B.A. treated plant* is given in figure 4. No leaf primordia are found, after the formation of the



Fig. 4. Cone shaped growingpoint of a tomatoplant treated with T.I.B.A. (magnification $80\times$).

5th leaf no more leaves are produced. Control plants produced 10 leaves in the same time. The growing point assumes the shape of a cone. This shape is kept for considerable time. The outside basal part of the cone has a normal epidermis with hairs and glands; the tip however lacks these organs and its colour suggests a meristematic tissue. It consists obviously of an irregular mass of cells. Axillary meristems are transformed into the same cone shaped structures (see figure 5).

So it is apparent that *T.I.B.A.* affects the growth relations in the meristems. One consequence of this activity is, that no leaf primordia are made.

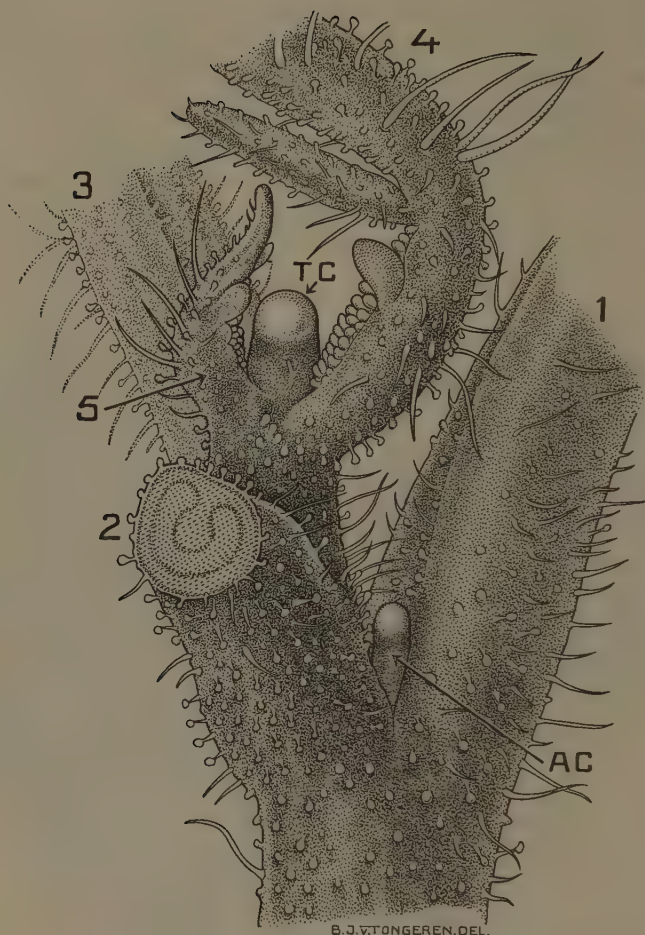


Fig. 5. Cone shaped terminal and axillary growingpoints of a tomatoplant, treated With T.I.B.A. (Magnification 20 $\times$).

TC — terminal cone.

AC — axillary cone.

6. *The influence of T.I.B.A. on unicellular organs (root hairs).*

Seeds of *Lepidium sativum* were germinated upon porous plates, permeated with a solution of T.I.B.A. (200 mg/l). The plates were placed (at an angle of 60°) in glass containers with the solution and closed with filter paper. The roots were growing against the plates and thus the root hairs are continuously in contact with the solution.

Microscopical investigation shows that in a root hair cell, growing under the influence of T.I.B.A., there is a great mass of cell wall material hanging in the inside of the cell at the tip (figure 6). Moreover the longitudinal growth has stopped. The material shows all cellulose reactions and has no birefringence. Thus it consists (in any case for the greater part) of cellulose and its presence indicates that the formation of cell wall material continues

under the influence of T.I.B.A. But the possibility to use this material for growth of the cell wall is absent, in other words, the material cannot be laid down to orderly arranged cellulose micels. Longitudinal growth of the cell is thus impossible, for it is a well known fact that the cell wall of a root-

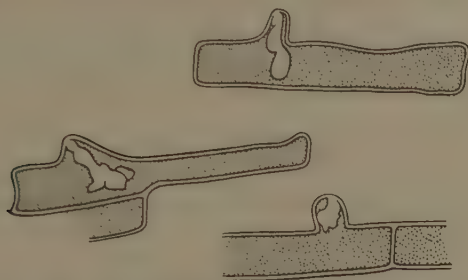


Fig. 6. Epidermal cells with roothair initials of *Lepidium sativum*. No longitudinal growth, formation of cellulose masses at the tip.

hair only grows at the utmost tip, only this part having a primary cell wall, whereas the adjoining parts have secondary cell wall layers. It is a striking fact that the cellulose mass is hanging at the tip, exactly at the point where in a normal hair growth takes place. Thus it is clear that *T.I.B.A. inhibits growth in length of a cell, but that the metabolic processes leading to the production of cellulose continue.*

7. Discussion.

What happens in a unicellular organ under the influence of T.I.B.A., namely cessation of longitudinal growth and continuation of the formation of cell wall material, can just as well take place in the cells of a meristem. This will lead not only to a decreased longitudinal growth, but also to very complicated growth relations, especially when not all cells are affected in the same degree. Moreover when there is an interference with the auxin, made in or present in the tip, the most various alterations of the shape of the cells and of the meristem can be expected. For the relation of longitudinal growth of a cell to growth in radial and tangential direction depends on auxin concentration (DIEHL, et al (1). The growing points of T.I.B.A. treated plants have a very slow (or no) longitudinal growth. [The cone, represented in figure 5, developed in the time an untreated control plant made 5 leaves and a flower bud.] So an interaction of T.I.B.A. with auxins is very probable.

T.I.B.A. as an anti-auxin has been found in other cases. DE WAARD and FLORSCHÜTZ (6) and GALSTON (2) stated that T.I.B.A. in a concentration of 5—50 mg/l, antagonized hetero-auxin activity in the standard *Avena* test. THIMANN and BONNER (3) found a growth stimulating effect of low concentrations of T.I.B.A. (together with auxins) and a growth retarding effect of high concentrations in three tests viz.: *Pisum* test, *Avena* standard test and *Avena* cylinder test. The stimulating effect could not be found by DE WAARD and FLORSCHÜTZ (6) in the standard *Avena* test. The con-

clusion, drawn by THIMANN and BONNER (3) from their experiments was: that T.I.B.A. possibly combines with the same substrate, with which auxins combine to bring about their growth promoting activity. The combination of T.I.B.A. with this substrate does not bring about growth, but when T.I.B.A. concentration is low, it leaves open a small number of spaces or active groups on the substrate with which auxin can combine. In this case a relatively small amount of auxin can bring about a disproportionate amount of growth.

As to the question of the florigenic properties of T.I.B.A., our observations yield no new views.

It was stated that T.I.B.A. caused alterations in the shape of the growing points and that these alterations were correlated with the inability to produce leaves and with the ability to produce flowers (in greater number).

Hence more floral primordia can be initiated under the influence of T.I.B.A.

When we assume the thesis "without florigen no flowers" the greater number of flowers can be caused by a stimulating effect on the "florigen" that is present already, respectively by the formation of a greater quantity of "florigen".

There is no indication that T.I.B.A. is a "florigen" itself or is the cause of *new* "florigen", as can be derived from the experiments with plants that were artificially kept vegetative (GALSTON (2)). The fact that inflorescences develop in the axils of lower leaves than usually, does not prove that they are "premature". In our experiments they never developed *before* an untreated plant, grown under the same conditions, has become flowering ripe, viz. had floral primordia.

But as long as "florigen" is a hypothetic substance, it is no use to discuss the question whether T.I.B.A. has "florigenic" capacity.

Summary.

1. T.I.B.A. causes unicellular organs (root hairs) to stop longitudinal growth, whereas production of cell wall material continues.
2. Growing points of tomato plants, treated with T.I.B.A., show an alteration of shape (cone). Correlated with this phenomenon is the inability to produce leaves and the ability to produce flowers.
3. In tomato plants treated with T.I.B.A. the terminal and axillary growing points (also of the cotyledons) can develop into inflorescences. The "abnormal" positions of these inflorescences are declared as correlation phenomena.
4. Tomato plants treated with T.I.B.A. can produce more flowers than untreated plants grown under the same conditions.

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Botany. — *A modified cylinder test of high sensitivity for growth substances.* (Preliminary note II.) By J. RIETSEMA *). (Communicated by Prof. V. J. KONINGSBERGER.)

(Communicated at the meeting of November 26, 1949.)

Introduction.

The quantities of growth substances present in natural sources are far too small to be estimated by chemical methods. Only for indole-3-acetic acid there are several sensitive colorimetric methods, which can be used for a minimal concentration as low as 5 mg/l (ALGEUS (1946)). However, some of these methods are not specific, while the specificity of others is open to question.

Compared with physiological methods these assays are very insensitive. Consequently small amounts of growth substances are determined with auxin deficient test objects.

Since WENT (1928) introduced the standard *Avena* test, several methods have been described. The most widely used are compiled in table 1. But as THIMANN (1948) states, many assay methods are no more used as assays but in the studies of the growth phenomena.

Each of the methods mentioned in table 1 has its pros and contras. We will restrict ourselves, however, to a discussion of the cylindertest, showing the following advantages over the standard *Avena* test:

1. It can be used in a large range of concentrations.
2. No constant humidity is required.
3. The growth substance needs not to be transported, but enters immediately into the growing cells.
4. Straight growth can be measured more exactly than curvatures for which the standard protractor is inaccurate especially at small reactions (KRAMER and WENT (1949)).

Points 1, 2 and 3 are shared by other immersion methods, which, however, depend on curvatures. The cylindertest has also some disadvantages as compared with the standard *Avena* test. The most important are:

- a. A relatively large amount of test solution is necessary for a measurable reaction (Vid. table 1, 4th column).
- b. It is less specific. Straight growth is, e.g., induced by sugars (SCHNEIDER (1938)) and the test is much more sensitive to ions than the *Avena* test is (WUHRMANN (1937), THIMANN and SCHNEIDER (1938)). Moreover, it is more sensitive to inhibiting substances.

The latter disadvantage, however, turns out into a distinct advantage in

*) This work has been carried out in the Botanical Laboratory of the State University, Utrecht, and has been supported by the "Stichting voor Zuiver Wetenschappelijk Onderzoek in Nederland".

TABLE 1.

Survey of assay methods for growth substances. Compilation from data given by HEYN (1936), WENT and THIMANN (1937), THIMANN (1948), and original papers. The term "critical" refers to the lowest limits.

Method	Author	Critical concentration mg/mm ³	Critical volume mm ³	Critical number of plants	Critical amount mg	Upper/lower concentration limit	Relation between growth or curvature and concentration
Standard <i>Avena</i> test	WENT, 1938 V. D. WEY, 1931	$5 \cdot 10^{-9}$ *)	$43.2 / 12$		$2 \cdot 10^{-7} / 12$	10	conc. (:) ° curvat.
Desecded <i>Avena</i> test	SKOOG, 1937	10^{-9} †)	$43.2 / 12$		$4 \cdot 10^{-8} / 12$	30	conc. (:) ° curvat.
<i>Cephalaria</i> test ‡)	SÖDING, 1937	10^{-8} — 10^{-11}	$80 / 10$		$8 \cdot 10^{-7}$ — $8 \cdot 10^{-10} / 10$		shape of activity curve variable
Quartered coleoptile test	THIMANN & SCHNEIDER, 1939 V. SANTEN, 1940	$3 \cdot 10^{-10}$ §)	$2000 / 4$ (8 curvat.)		$6 \cdot 10^{-7} / 4$	$1/3 \cdot 10^{-4}$	approx. log conc. (:) ° curvat.
Pea test	WENT, 1934 V. OVERBEEK & WENT, 1937	$1.5 \cdot 10^{-8}$	$20000 / 8$ (16 curvat.)		$3 \cdot 10^{-4} / 20$	10^3	approx. log conc. (:) ° curvat.
Cylinder test	J. BONNER, 1933 SCHNEIDER, 1938 V. SANTEN, 1938	$5 \cdot 10^{-9}$	$40000 / 20$ **)		$2 \cdot 10^{-4} / 20$	10^2	approx. log conc. (:) % growth
Modified cylinder test		10^{-11}	$2000 / 20$		$2 \cdot 10^{-8} / 20$	10^4	$\log V = 1/n \log C + k$

*) In the Utrecht Laboratory a concentration of about $5 \cdot 10^{-9}$ gives a curvature of 1°. The dimensions of the agar block are 6—8—0.9 mm.

†) According to THIMANN the method is 3—5 times as sensitive as the standard test.

‡) The method is not suitable for quantitative work.

§) The critical concentration is the lowest concentration giving a significant inward curvature (THIMANN and SCHNEIDER, 1939).

**) Critical volume after SCHNEIDER (1938).

the study of synthetic growth substances, which are inactive in the *Avena* test, because they are not transported (VAN OVERBEEK and WENT (1937), VELDSTRA (1944)).

Owing to the latter properties the cylindertest was used relatively seldom as an assay method, but mainly as an indicator in studying growth phenomena.

By applying a modification, however, we succeeded to improve the sensitivity of the cylindertest to such a degree, that it becomes possible to estimate extremely small quantities of growth substances. Moreover, a mathematical relation between concentration and effect has been found.

Material and Method.

When not mentioned otherwise the experiments were carried out in the same way as described in a previous paper (RIETSEMA (1949)): $1\frac{1}{2}$ mm sections, 5— $6\frac{1}{2}$ mm from the tip of three days old coleoptiles, were mounted on a slide by means of a narrow strip of pure vaseline. The slides were placed into a petri dish with 100 ml solution. The length of the sections was measured under a microscope. During the experiments the solutions were aerated intensely. The measurements were made in orange light; before and after measuring the sections remained in complete darkness.

The growth of zones of intact coleoptiles was measured by placing the seedlings in front of a horizontal microscope. The segments were marked with tiny dots of India ink, applied with a fine brush.

Experimental.

A. Effect of indole-3-acetic acid on growth and growth rate.

According to the usual computation methods the difference between the means of two samples will be more significant as the standard error of the difference is small (PATERSON (1939)). Consequently the difference between two experiments will be influenced favourably as the magnitude of the outcome of the control experiment can be kept relatively small.

Examination of the growth of the *Avena* coleoptile sections shows that the growth rate is very high during the first 4 hours after immersing the sections into the solution. After about 5 hours the growth rate decreases and remains very low and almost constant during the next 20 hours. The relation between growth and pH is established after $4\frac{1}{2}$ hours and remains unchanged during the next 20 hours, the length increasing only slightly, indifferent from pH (VAN SANTEN (1940), p. 21, fig. 3). In our experiments for low indole-3-acetic acid concentrations the difference in growth with the control did not increase any more after 6 hours.

These facts are unfavourable for quantitative work, as the control value is high as compared with the effect of indole-3-acetic acid. For that reason the standard *Avena* test is advantageous as the curvatures are independent of the residual growth and the controls consequently superfluous (WENT (1934)).

So to make the cylindertest usable as a quantitative assay method for small amounts of growth substances the growth rate of the control sections must be kept low.

To reach this aim we considered the possibility to add growth substances to the sections after the decrease of the high initial growth rate to a low constant level. In table 2 the results of three experiments are summarized

TABLE 2.

Growth and growth rate of *Avena* coleoptile sections after addition of indole-3-acetic acid, 0.1 mg/l, at different moments after the beginning of the experiment. The figures given are of three experiments. Age of the coleoptiles before sectioning in I and II 72 hours, in III 96 hours. pH of the buffer solution in I and III 5.9 and 5.5 in II.

Time between preparation and addition hrs			Growth rate after addition ‰ elongation/hr			Growth rate of control at the same time ‰ elongation/hr			Differences			Differences in ‰ of the control			Growth in ‰ elongation after		
															36	40	36 hrs
I	II	III	I	II	III	I	II	III	I	II	III	I	II	III	I	II	III
0	0	0	6.6	4.3	3.3	2.2	2.0	2.0	4.4	2.3	1.3	200	87	65	41	32	25
3	3	2.5	4.5	1.7	(0.7)	0.8	0.4	0.3	3.7	1.3	(0.4)	460	325	(100)	55	27	(14)
6	6	6	4.3	1.8	0.8	0.6	0.2	0.1	3.7	1.6	0.7	620	800	700	62	34	20
9	9.5	9.5	3.0	1.9	0.7	0.2	0.1	0.03	2.3	1.8	0.7	1400	1800	2330	58	28	17.5
12.5	13.5	13.5	3.1	1.8	0.7	0.2	0.1	0.03	2.9	1.7	0.7	1450	1700	2330	59	22	14
17	18	17	4.0	1.7	0.8	0.2	0.1	0.03	3.8	1.6	0.8	1900	1600	2670	43	30	14
21	23	23.5	4.0	2.0	0.8	0.2	0.1	0.03	3.8	1.9	0.8	1900	1900	2670	41	31	14
26	29	28	3.3	0.9	0.6	0.2	0.1	0.03	3.1	0.8	0.6	1550	800	2000	31	23	13

of which exp. I is shown in fig. 1. Indole-3-acetic acid, 0.1 mg/l, was added to the buffer solution, only the moment of addition of the growth substance is varied. It must be noted that the total growth after 36 hours is optimal when the growth substance is added 6 hours after immersing the sections into the solution.

There is a linear relation between growth and time during the first hours after the addition of indole-3-acetic acid. This is true except when heteroauxin is added at the beginning of the experiment. After 10—18 hours the growth rate decreases rapidly to almost zero. This phenomenon (cf. the shape of the curves in fig. 1) allows to suppose a limiting factor, but presumably not indole-3-acetic acid. If the added growth substance were the factor limiting growth, the period between addition of heteroauxin and the decrease of the growth rate should be constant. As the length of this period decreases as the time between sectioning and addition of the growth substance is longer, it is suggested that a factor present inside the cells and necessary for growth is used up in metabolic processes. The nature of this substrate is unknown for the present.

As is clearly shown in fig. 1 and table 2 the growth rate immediately after addition of indole-3-acetic acid is independent of the length of stay of the

sections in the buffer solution. The growth lines in fig. 1 of the growth induced by heteroauxin are equidistant.

There are, however, two exceptions. The growth rate has its maximum

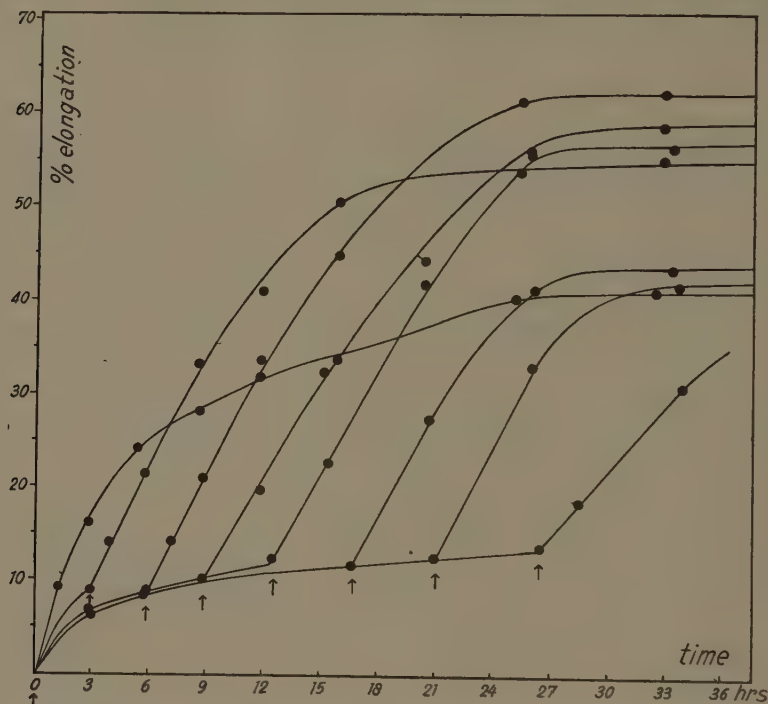


Fig. 1. Growth of *Avena* coleoptile sections in a 0.01 m phosphate buffer solution, pH 5.9. Indole-3-acetic acid, 0.1 mg/l, added at different moments after the beginning of the experiment, indicated by arrows.

when the growth substance is added at the beginning. Also the control is high at this moment, but the difference now is even slightly larger than the differences at later moments of addition. So the effect of indole-3-acetic acid on the growth rate in % elongation per hour is largest when the substance is added as the experiment begins. Compared with the control, however, it is optimal when added between 6 and 26 hours after the first measuring of the sections (cf. table 2, 4th and 5th column).

The second exception is made when the sections have grown for 26 hours in the solution. Adding of the growth substance now results in a smaller increase of the growth rate. Probably new factors interfere with growth due to the long stay in the liquid.

It is evident that while within certain limits there is no change of the effect of heteroauxin with time, there is no "aging". According to WENT (1935) "aging" means a stiffening of the cell wall by cell wall formation exceeding the rate of elongation. In a buffer solution no new material can be supplied for the cell wall, in consequence we may expect no "aging". With "aging" the growth rate should decrease when the period before

addition grows longer. From the 5th column of table 2 we can see that the increase in growth rate as % of that of the control rises gradually and is optimal after 9 hours.

These results make it possible to estimate low concentrations of indole-3-acetic acid. The requirement of a low control value is answered by waiting 6—9 hours before addition. A 6 hour period was chosen because otherwise the duration of an experiment becomes too long.

B. Sensitivity.

To investigate the sensitivity of the method a number of experiments has been performed with extremely low concentrations of indole-3-acetic acid added after 6 hours. Amounts down to 10^{-5} mg/l turned out to be effective. The results of two experiments are given in table 3. The growth

TABLE 3.

Comparison of the effects of indole-3-acetic acid on growth and growth rate of *Avena* coleoptile sections. Addition of the growth substance 0 and 6 hours after preparation of the sections in distilled water. The figures 0 and 6 above each column refer to the time of addition.

Indole-3-acetic acid concentration mg/l	Growth rate after the addition % elongation/hr	Differences with control	Probability of the differences*)	Growth after 24 hours % elongation		Differences with control		Probability of the differences*)	
				0	6	0	6	0	6
0	0.86 ± 0.14			16.5 ± 0.6	16.5 ± 0.6				
10^{-2}	3.27 ± 0.18	2.41 ± 0.23	1	22.9 ± 1.1	33.6 ± 1.2	6.4 ± 1.3	17.1 ± 1.3	1	1
10^{-3}	2.07 ± 0.15	1.21 ± 0.20	1	20.5 ± 0.7	25.3 ± 1.1	4.6 ± 0.9	8.8 ± 1.3	1	1
10^{-4}	1.48 ± 0.11	0.62 ± 0.18	1	19.0 ± 0.8	21.3 ± 1.2	2.5 ± 1.5	4.8 ± 1.3	0.91	1
10^{-5}	1.20 ± 0.17	0.34 ± 0.22	0.88	19.5 ± 0.8	20.5 ± 1.2	3.0 ± 1.0	4.0 ± 1.3	1	1
0	0.77 ± 0.13			20.7 ± 1.4	20.7 ± 1.4				
10^{-2}	2.85 ± 0.22	2.08 ± 0.25	1	26.5 ± 1.9	28.3 ± 1.1	5.8 ± 2.5	7.6 ± 1.7	0.98	1
10^{-3}	2.09 ± 0.24	1.32 ± 0.27	1	22.5 ± 1.2	25.4 ± 1.1	1.8 ± 1.8	4.7 ± 1.7	0.68	0.99
10^{-4}	1.50 ± 0.12	0.73 ± 0.18	1	22.5 ± 1.1	21.4 ± 1.1	1.8 ± 1.7	0.7 ± 1.7	0.71	0.32
10^{-5}	1.20 ± 0.13	0.43 ± 0.19	0.98	21.5 ± 1.0	22.0 ± 1.1	0.8 ± 1.7	1.3 ± 1.7	0.36	0.55

*) Probability that the difference is due to the treatment, according to FISHER's χ table [PATERSON (1939) p. 247].

of coleoptile sections in distilled water with and without addition of indole-3-acetic acid after 0 and 6 hours has been compared.

A comparison of the elongation after 24 hours of the subsequent series shows that addition after 6 hours gives best results (table 3). 10^{-5} and 10^{-4} mg/l increase the growth significantly in the first experiment but insignificantly in the second. The increase in growth, however, is very small.

More important is a comparison of the growth rates. As is shown in table 2 the growth rate is increased immediately after the addition of indole-3-acetic acid at 0 hour, but the increase is only a small percentage of the growth rate of the control. Moreover, the initial growth rate usually is varying considerably, making the evaluation of the results even more difficult.

On the contrary the growth rate after 6 hours is very low and the variability between the series small. As table 3 shows, the growth rate of the sections increases when higher concentrations of indole-3-acetic acid are added. The effect of 10^{-5} mg/l in the first experiment is nearly significant and in the second one it is real, as it is within the 5 % range of probability.

C. Quantitative relation.

For quantitative work it is desirable that the relation between concentration and effect can be expressed mathematically to allow inter- and extrapolation.

The log of the increase in growth rate plotted against the log of the concentration gives in almost every case a straight line. Above a certain concentration the growth rate does not increase further and decreases again at very high concentrations.

So the relation between concentration of heteroauxin added after 6 hours and the increase in growth rate it gives can thus be expressed:

$$\log V = \log C^{1/n} + k$$

in which V is the increase in growth rate, C the concentration in mg/l, k and n constants, characteristic for the experiment.

This expression is the same as FREUNDLICH's for adsorption phenomena. V must be replaced by x/m , x being the amount of adsorbed substance, m the quantity of adsorbent.

The similarity of the two expressions tends to suggest an adsorption phenomenon taking part in the growth processes. For the present all proof is lacking for such a conclusion. Experiments dealing with this subject are in progress.

For high concentrations the formula is no more valid, in most cases the highest usable concentration is 0.1 mg/l indole-3-acetic acid.

Fig. 2 gives results of 4 experiments showing the linearity of the relation.

Although two important improvements of the method have been described, enhanced sensitivity and a linear relationship, the disadvantage of the large amount of liquid necessary still remains. As SCHNEIDER (1938) showed, it is possible to perform the test with 2 ml solution and without aeration (cf. J. BONNER (1933)).

From table 1 it is clear that the critical amount with regard to the standard and deseeded *Avena* test has been reduced to 1/10 and $\frac{1}{2}$

respectively. Moreover large concentrations can be estimated without serial dilutions. The objection of the lacking specificity remains. For the

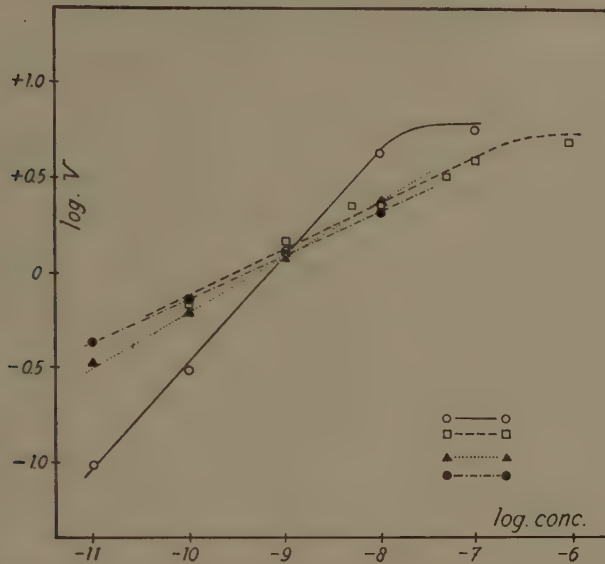


Fig. 2. Relation between the concentration of indole-3-acetic acid and the increase in growth rate of *Avena* coleoptile sections. The growth substance was added 6 hours after the beginning of the experiment.

present it may be sufficient to state that sugars do not cause an immediate increase in growth rate as growth substances do.

The simplicity of the method, not requiring much routine, may be an advantage.

Discussion.

It has been stated that the initial growth rate after submerging the sections is high and variable. The growth rate therefore cannot be a measure for the concentration when indole-3-acetic acid is added at the moment of immersion. The increased sensitivity after 6 hours has been explained by the reduced growth rate of the control. A point of interest remains why the growth is not optimal when indole-3-acetic acid is added in the beginning and why the initial growth rate is so high. These problems will be considered next.

The initial elongation of the sections is large compared with the total growth after 24 hours. If the growth of corresponding zones in intact seedlings is compared with the growth of immersed sections, in both cases the growth rate is about the same during the first 30 minutes. Sometimes the isolated sections grow even more in $\frac{1}{2}$ hour. The contrary is the case when instead of $1\frac{1}{2}$ mm sections 3 mm sections are compared. In this case the intact coleoptiles grow always faster (cf. table 4). The 3 mm sections have grown more after 24 hours than the $1\frac{1}{2}$ mm sections.

TABLE 4.

Growth of *Avena* coleoptile sections of different length compared with the growth of corresponding zones in coleoptiles of intact seedlings. The figures above each column indicate the zone in mm from the tip from which the sections were cut. Growth in % elongation, average of 20 sections.

	Isolated sections				Intact seedlings		
Time hr	4-5½	5½-7	Mean	4-7	4-5½	5½-7	Mean
1	7.2	5.1	6.2		6.4	5.3	5.9
2	8.7	6.4	7.6		14.1	12.3	13.2
½	5.1	3.5	4.3	2.1			
1	7.2	5.5	6.4	3.6	9.4	8.5	8.9
2	8.8	7.0	7.9	6.3			
3½	11.3	8.7	10.0	8.5	29.0	28.8	29.9
½	4.0	2.8	3.4	1.5			
1	6.1	4.6	5.3	3.2	11.8	5.9	8.6
2	7.8	6.6	7.2	5.4	31.7	21.8	26.8
24	14.2	11.4	12.8	15.2			
24	16.0	13.1	14.6	15.5			

Isolated sections are cut off from every supply coming from the tip or the seed. The great initial elongation of the short sections therefore must be due to the shortness of the sections and is of less importance when the cylinders are longer.

VERY and BURKHOLDER (1937) have shown that in a coleoptile cell-divisions only take place in the early stages of growth so that when the coleoptile is 10 mm long the number of cells does not increase any more. Cell counts and measurements on coleoptiles showed that the number of epidermal cells is smallest and the number of subepidermal cells is largest. Each successive inner layer has fewer cells. For a 9.83 mm coleoptile the figures given are: 31, 184, 163, 148, and 119 for the inner epidermis (cf. VERY and BURKHOLDER, table 3, p. 8). From our measurements on 10.3 mm sections, 4-7 mm from the tip, a mean of 1.37 epidermal cell per mm was found or 4.11 cell per section.

A great number of cells is damaged by sectioning. The number of injured cells will be relatively larger when the sections are shorter. In the case of 4.11 cells per 3 mm we can expect from computation 3.2 intact cells on 3 mm and 1.1 intact cells in sections of 1½ mm. For the other cell-layers sectioning will have less influence on the number of intact cells per mm because they are much shorter. As long and short cylinders only differ in the number of intact epidermal cells per mm, this probably will be the cause of the difference in initial growth (table 4). The wounded surface of the short sections is relatively large and consequently should

cause a smaller elongation. So this factor can be excluded though in curvature tests it is important (WENT (1939)).

In intact tissue the different layers have a tendency to grow at different rates. Mostly the epidermis will be under tension and the inner layers under compression (THIMANN and SCHNEIDER (1938)). The tissue tension caused by the different growth rates is shown clearly when a coleoptile is slit longitudinally. The two strips show outward curvatures. So by compressing a cell the turgor tension will increase and in the intact tissue will be in equilibrium with the cell wall pressure and the tissue tension. After slitting the tissue tension has nearly been eliminated and the strips curve outward.

In sections, however, the tissue tension is not eliminated. As in short sections the number of epidermal cells with regard to the number of cells in the inner layers is relatively small, the compression force of the inner layers will be low. As the tissue tension decreases, the equilibrium will be disturbed and after submerging the inner layers will take up water and increase in length. The osmotic water uptake will be larger as the number of epidermal cells is low.

So according to the above the initial elongation is mainly due to a passive water uptake and not to actual growth. As this process proceeds quickly, small differences in time between submerging and measuring may give relatively great errors. Growth substance added at this moment may interfere with this water uptake. Consequently we do not observe an influence on growth only. By waiting 6 hours only the growth promoting action of the added growth substance is left.

Summary.

A modification of the cylinder test for growth substances is described in which the growth substance is added 6 hours after the immersion of the sections into the liquid.

By measuring the growth rate after the addition it is possible to estimate indole-3-acetic acid concentrations down to 10^{-5} mg/l. A volume of 2 ml solution is enough for the test; so an amount of $2 \cdot 10^{-8}$ mg can be estimated (vid. table 1).

By plotting the log of the concentration against the log of the increase in growth rate a straight line is obtained, fit for intrapolation.

It is highly probable that the low sensitivity of the original test can be ascribed to an osmotic water uptake during the first hours of the growth which interferes with the action of the growth substance. This may be the cause that the growth after 24 hours is larger when the growth substance is added after 6 hours.

Acknowledgements.

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Zoology. — *The placentation of Tupaia javanica*. By C. J. VAN DER HORST, (Hubrecht Embryological Laboratory and University of the Witwatersrand.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of October 29, 1949.)

In 1899 HUBRECHT published a description of the placenta of *Tupaia javanica*. Although small additions to this description were made by DE LANGE and NIERSTRASZ (1932) and again by DE LANGE (1933), these authors fully accepted HUBRECHT's explanation of the placental structure. At the time of HUBRECHT's publication, GROSSER's (1909) classification of the various placentas of mammals was still unknown, but in 1933 DE LANGE classified the placenta of *Tupaia* as being haemo-chorial and this was in agreement with HUBRECHT's description.

When I had an opportunity to visit the HUBRECHT Laboratory I was anxious to study the placenta of *Tupaia*. Not only were all facilities for doing so given to me, but I was presented also with a number of uteri of *Tupaia* in various stages of pregnancy. These were later on sectioned in Johannesburg, where I could study them more at leisure.

When studying the placentation of *Elephantulus* (VAN DER HORST, 1950) it occurred to me that, from a morphological point of view, the placenta of *Tupaia* could be different from that of *Elephantulus*. And, as the placenta of *Elephantulus* fundamentally corresponds to that of Insectivores like *Erinaceus* and the *Soricidae* and also to that of the Primates like *Tarsius* and the monkeys, it would be of even more interest to establish the relationship of the placenta of *Tupaia*. In these Insectivores and Primates the placenta is either mesometrial or antimesometrial, or either dorsal or ventral in position and, what is even more important, it is formed either at the same side as the embryonic node or at the opposite side. If it is remembered that in monkeys two placentas are often present at opposite sides and that *Elephantulus*, with the placenta and embryo both at the mesometrial pole, forms a rudimentary placenta opposite the permanent one, then this difference appears to be of no great importance. In *Tupaia*, however, the embryo is directed towards the antimesometrial side, but the two placentas are formed in the lateral walls of the uterus. This might be an indication that the placenta of *Tupaia* is a structure totally different from that of other Insectivores or Primates. As we will see in the following pages, it really is.

As the results of my observations deviate in several, and sometimes important, respects from HUBRECHT's description, I will endeavour to give a complete description of the whole process of placentation in *Tupaia* as far as the available material allows, indicating, by the way, where I cannot agree with HUBRECHT.

The youngest stage, at my disposal, contains an embryo in the stage of a bilaminar blastocyst, lying free in the uterine lumen near the place where soon it would have fixed itself to the lateral walls of the uterus (fig. 1). Here, as HUBRECHT also mentions, the implantation sites are

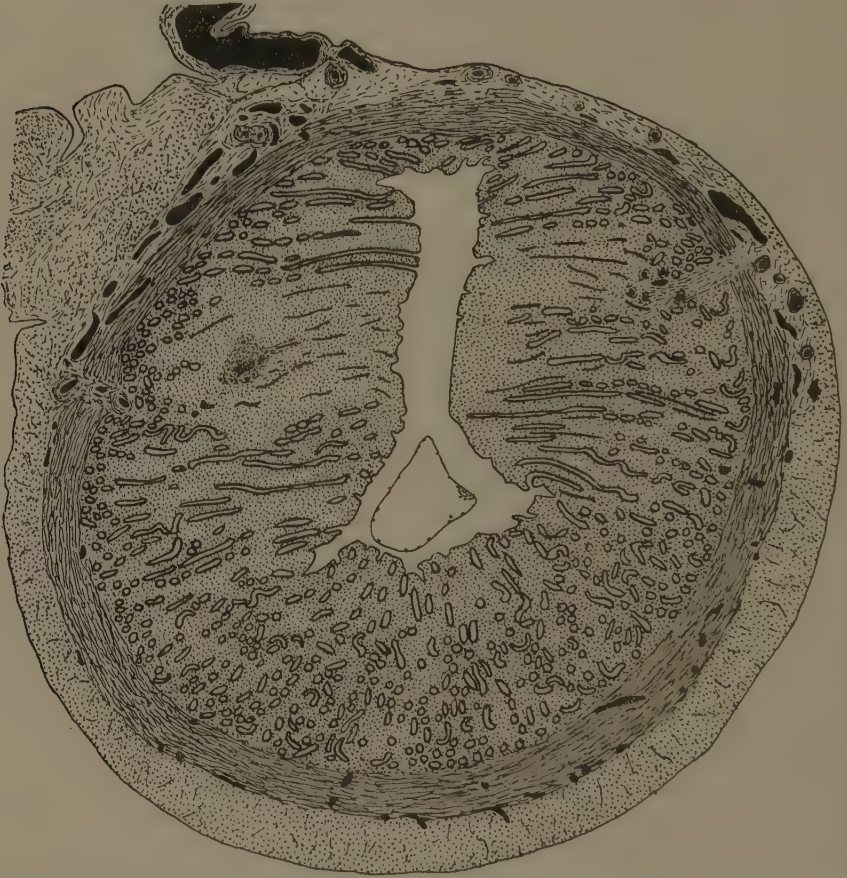


Fig. 1. Cross section of the uterus with an embryo in the early blastocyst stage, before it has attached itself to the uterine wall. $\times 30$.

already clearly indicated before the embryo comes in direct contact with them. It is, therefore, likely that this is the only place along the length of each uterine horn where an embryo can implant. In other words, in *Tupaia*, like in *Elephantulus*, implantation is strictly localised and orientated. The implantation sites are richly vascularised; the circular musculature is even interrupted where the bloodvessels enter the endometrium. These vessels, the arteries as well as the veins, branch repeatedly after their entrance into the endometrium.

An obvious characteristic of the implantation sites is the absence of glands. HUBRECHT says that, here near the surface, the glands have become reduced (*rückgebildet*) and that they remain only in the deeper part of the mucosa. This statement is not quite correct. Along the whole circum-

ference of the uterus the glands are very numerous in the deeper part of the mucosa facing the circular musculature. These glands usually continue as irregularly curved tubes through the more superficial region of the mucosa, to open anywhere in the uterine lumen. This, at the level of implantation, is particularly clear in the antimesometrial wall (fig. 1). In the lateral walls, however, the glandular tubes are deflected and form, on the whole, a simple curve round the implantation sites. Therefore, the glands here can often be seen in sections as long and nearly straight tubes surrounding the implantation area. The glands, seen below the implantation site, therefore are not reduced, nor do they end blindly, but they go in a wide swoop round the implantation area on their way to the uterine lumen.

When the embryo has developed 12 pairs of somites the two lateral placentas are well established (fig. 2) and their structure is revealed (fig. 3). Only the very first beginning of the allantois may be discerned. The yolksac, which is now trilaminar, is alone in contact with the inner



Fig. 2. Cross section of the uterus containing an embryo with 12 somites. $\times 20$.

surface of the maternal part of the placenta and extends over the whole of it. Facing the muscularis, all round the uterus, the stroma consists of a layer of rather dense fibrous tissue in which the coiled deeper parts of the glands are embedded. Underneath the placenta there are relatively few



Fig. 3. Part of a cross section of the placenta of the same specimen as shown in fig. 2. The section is the next one to that shown in fig. 2. $\times 100$.

glands, which may be caused by the stretching of the uterine wall. These glands can be traced to the circumference of the circular placentas, where they open into the uterine lumen. Underneath the placenta, at the inside of the layer of glands, the fibres become very delicate and few in number; in the rest of the uterine wall the thicker layer of glands extends nearly to the uterine epithelium.

At this stage the placentas are very obvious structures. The maternal part, called by DE LANGE (1933) the "Plazentarscheiben", is derived from the vascular tissue, without glands, the "Plazentarleisten" of DE LANGE, that can be observed before the attachment of the embryo (fig. 1). For the greater part these plaques consist of a very dense tissue, crowded with nuclei. The number of nuclei steadily increases as is shown by the great number of mitotic figures in the deeper region. It is obvious that a vigorous growth of this layer takes place. However, whereas it grows at the peripheral side, this layer is dissolved and digested on the inside. About in the middle of this layer the darkly stained nuclei congregate in larger or smaller clusters. The individual nuclei can still be discerned at this level, but more to the inside the nuclei break up and only a dark irregular mass of chromatic substance is left. Between these clusters of nuclei, or what remains of them, the tissue is very open with only a few scattered and apparently healthy nuclei, many bloodvessels are embedded in this tissue, the trophospongia of HUBRECHT.

On the inside this placenta is covered by the trilaminar yolk sac. The trophoblast can be seen penetrating into the maternal tissue with short blunt protruberances, but the mesoderm, richly supplied with omphaloidean vessels, does not penetrate at all.

Concerning the bloodvessels, many arteries can be seen penetrating into

the placental disc from the muscularis. These arteries, giving off several branches at all levels into the dense layer, pursue on the whole a straight course, until they reach the necrotic mass near the inner aspect of the placenta. Here they branch up into smaller vessels. A rich network of small vessels can be observed in the necrotic tissue, where they are more obvious, and in the densely nucleated tissue, where these vessels can be seen only with some difficulty. The larger veins assemble the maternal blood near the peripheral surface. Therefore, on the whole the maternal blood flows through the placenta from the inner, embryonic, side towards the periphery. The straight course of these arteries, and the fact that they give off small side branches at any level, shows that they are certainly not coiled arteries. As coiled arteries are so conspicuous a feature in the uterus of *Elephantulus* during early pregnancy, their absence in *Tupaia* is noteworthy, the more so as coiled arteries also occur in the *Centetidae* and in *Erinaceus* amongst the *Insectivores* and in monkeys, apes and man amongst the *Primates*.

In the next stage available, the length of the embryo, unfortunately, could not be measured. The allantois is in the process of spreading out over the inner surface of the placenta, here replacing the yolk sac. Here and there the allantoic mesoderm, carrying embryonic bloodvessels, starts to penetrate into the placenta as blunt processes (fig. 4). It is obvious



Fig. 4. Part of a cross section of the placenta at the time when the allantois is spreading out over it. $\times 100$.

that at this stage the densely nucleated tissue, forming the peripheral part of the placenta, has considerably decreased in thickness as compared with the earlier stage, and that notwithstanding the vigorous mitotic activity of the earlier stage. Even at this stage several nuclear divisions could be observed, indicating that active growth of this layer continues. But at a greater rate than it is formed, this layer disintegrates at its inner aspect. The dark masses of breaking down nuclei with the open spaces between them, now occupy about the middle third of the placenta. The trophoblast has penetrated deeper and evidently absorbs the disintegrating mass. Often it is difficult to define the surface of the trophoblast exactly, because in

the first place here and there some healthy maternal cells are left and in the second place the trophoblast is often less dense where it penetrates deepest (fig. 5).

HUBRECHT also described this increasing thickness of the trophoblast at the expense of the maternal trophospongia. According to him the

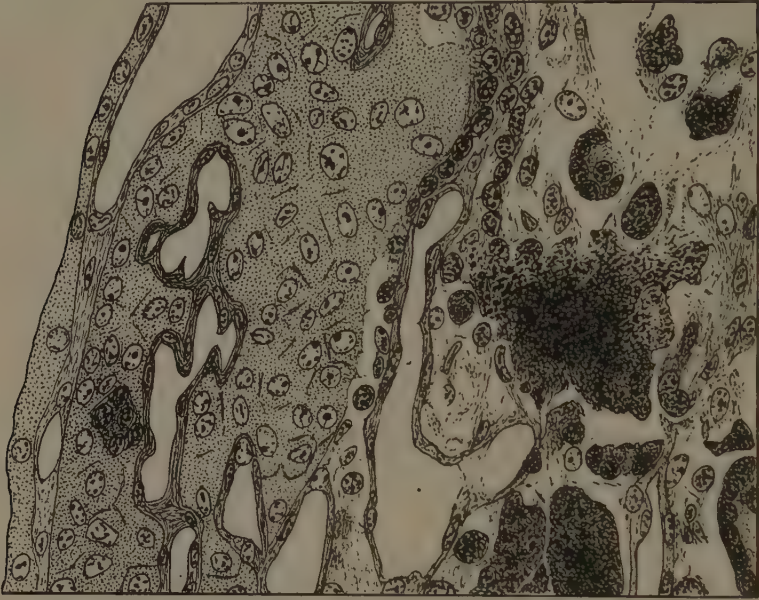


Fig. 5. A small part of the same placenta as shown in fig. 4, near the edge of the allantois. $\times 500$.

deepest penetrations are formed by plasmoditrophoblast, whereas the cytotrophoblast is found nearer the embryonic side. It may be true that at places, where the trophoblast has penetrated deepest, the cell membranes are absent; if so then these areas are of very limited dimensions and only transitory in nature. The greater part of the trophoblast definitely is cellular (fig. 5). On the other hand, I could not observe amitotic nuclear division in the trophoblast, which was mentioned by HUBRECHT. All nuclear divisions in the trophoblast are mitotic ones. According to HUBRECHT the trophoblast cells, upon making contact with the uterine epithelium, increase in size and their nuclei become larger; he also speaks about megalocaryocytes. The trophoblastic nuclei are often somewhat larger than the nuclei of the maternal tissue, but giant cells, which are so obvious in *Elephantulus*, as well as in several other mammals, do not occur in *Tupaia* in neither the maternal nor the embryonic tissues and at no stage of the development.

The process of penetration of the trophoblast continues in the following stages that could be studied. When the embryo is 7.14 mm long, the trophoblast, well supplied now with embryonic bloodvessels, occupies more than half the thickness of the placenta. Only a thin layer of the densely

nucleated maternal tissue is left at the outer surface, otherwise the whole peripheral part of the placenta consists of the necrotic trophospongia. In the next stage the embryo, in sections, is 11.34 mm long. Some isolated patches of the healthy maternal tissue are left at the periphery of the placenta and the trophospongia is reduced to a narrow layer. Otherwise the whole placenta consists of trophoblastic tissue with a network of allantoic vessels. When the embryo is 17.40 mm long, hardly any necrotic trophospongia is left.

It may be worth mentioning, by the way, that the embryonic erythrocytes are still nucleated when the embryo is 11.34 mm long, but that they have lost their nuclei at the 17.40 mm stage. There may be some relation between the final establishment of the placenta and the loss of these nuclei.

Of major importance is the fate of the maternal bloodvessels in the placenta. HUBRECHT says that the numerous maternal capillaries become surrounded by the proliferating trophoblast that penetrates into the trophospongia. After some time the endothelium of these maternal vessels should disintegrate, as was described by NOLF in *Vespertilio*. In this way the maternal blood should penetrate into the system of lacunae inside the trophoblast. DE LANGE (1933) accordingly classifies the placenta of *Tupaia* as a haemo-chorial one. It has been mentioned already that the placental arteries, upon leaving the muscularis, penetrate, in a nearly straight course, through the whole thickness of the placenta, although side branches are given off at different levels. This fact considerably facilitates the observation of these maternal vessels. It is easy to trace the course of an artery from the muscularis to deep in the placenta. As can be expected the endothelium is present in the densely nucleated, healthy, zone of the placenta and it is continuous with the endothelium of the same artery in the muscularis. In the decaying trophospongia the necrosis affects the densely nucleated tissue only, but the endothelium of the vessels is left intact (fig. 5). The endothelial nuclei have a healthy appearance. This endothelium again is continuous with that of the main arteries that penetrate into the placenta. Even the cells, immediately surrounding the side branches and further vessels of a capillary size, prove to be more resistant than the main mass of maternal tissue, although the nuclei become more pycnotic upon the approach of the trophoblast, and finally these cells succumb to the digestive influence of the trophoblast. But the endothelium of the maternal vessels remains intact even when these vessels are finally completely surrounded by the trophoblast (fig. 5). Even at later stages, when the densely nucleated tissue and finally the trophospongia have completely disappeared, the continuity of the endothelium of the radial arteries with that of the capillary network could be established without any doubt (fig. 6). Therefore, the placenta of *Tupaia* is not of the haemo-chorial but of the endothelio-chorial type, according to GROSSER's classification.

The dual placenta of *Tupaia* is different from that of *Elephantulus* and

of Insectivores like *Erinaceus* and the *Soricidae*, on the one hand, as well as from the placenta of *Tarsius* and the monkeys, on the other hand, in several respects. It differs in its general position, being located in the lateral walls of the uterus, whereas the embryo itself faces the antimesomet-



Fig. 6. A small part of the placenta associated with an embryo, at least 28 mm. long. A placental artery is shown in the centre. The maternal blood is black, the embryonic blood white. $\times 200$.

rial pole. It differs in its histological structure, being an endothelio-chorial placenta. And finally the whole mechanism of feeding the embryo is different. In *Elephantulus* and the other Insectivores and Primates, enumerated above, the embryo is fed by an exchange of substances between the maternal and embryonic blood. In *Tupaia* the mother builds up a highly nutritive substance in the form of the densely nucleated tissue and this is digested by the embryo. There will be, of course, an exchange between the embryonic and maternal blood as well, but one gets the impression that the decaying trophospongia is the main source of food for the developing embryo.

It is generally accepted nowadays that *Tupaia* is closely related to the Primates, in particular to the Lemuroidea. The systematic position of the *Macroscelididae*, however, is held to be more dubious; if they are supposed to be related at all to the Primates, then this should be through their affinities to *Tupaia*. I discussed the pros and cons of this taxonomic problem in more detail elsewhere (VAN DER HORST, 1950). It is remarkable that, concerning the placenta in all its aspects, *Elephantulus* closely agrees with the Primates. In particular the early placenta of *Tarsius* structurally is hardly different from that of *Elephantulus*, although *Tarsius* has the secondary placenta of the monkeys and *Elephantulus* the primary one with only a rudiment of the secondary. This haemo-chorial placenta is a characteristic common to several Insectivores and Primates. *Tupaia* has lost this placenta and in this respect it agrees with the Lemuroidea. But *Tupaia* differs from the Lemuroidea in having a double endothelio-chorial placenta. The latter have a well-developed epithelio-chorial placenta with often complicated interdigitations of the maternal and embryonic tissues.

In *Tupaia* the uterine glands remain functional at least for a long time,

if not during the whole period of pregnancy. In the oldest stage available, with an embryo at least 28 mm long, the glands do not show any sign of diminished activity and a capillary network can often be seen surrounding them. In *Elephantulus*, at this stage, the glands have long since disappeared. In the Lemuroidea, with their epithelio-chorial placenta, the growing embryo is fed by glandular secretion. In *Tupaia* the uterine glands also feed the embryo, although they supply only part of the necessary nutriment. But in contrast to the Lemuroidea, no interdigitations of the uterine and embryonic tissues have been elaborated; both the uterine epithelium and the trophoblast have a smooth surface where they are in close approximation to each other. This contact can only be considered as an epithelio-chorial placenta of a primitive type. It can be added that it is a yolk-sac placenta, as is clearly shown by the layer of very large entodermal cells that are separated from the trophoblast by a solid mesoderm layer. In the area of the endothelio-chorial placentas the yolk-sac is replaced by the allantois, but the latter does not extend further, nor does even the extra-embryonic coelom penetrate between the entoderm of the yolk-sac and the trophoblast in the region outside the endothelio-chorial placentas.

It may be concluded that *Tupaia* lost the original haemo-chorial placenta found in other Insectivores and Primates. The nutrition of the embryo by the uterine glands gained in importance as compared with *Elephantulus* but it did not reach the same level of perfection as in the Lemuroidea, where the uterine glands are the only source of food for the embryo. In *Tupaia* a double endothelio-chorial placenta has been evolved instead, a type of placenta which otherwise does not occur in the Insectivores or Primates.

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Zoology. — *The changes in the cytology of the adrenals of Rana esculenta following hypophysectomy.* By J. W. SLUITER, J. C. A. MIGHORST and G. J. VAN OORDT. (Zoological Laboratory, Dept. of Endocrinology, University of Utrecht.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of November 26, 1949.)

1. Introduction.

Following investigations into the regulation of spermatogenesis in the frog, the influence of the pituitary on the adrenals of the green frog, *Rana esculenta*, has now been studied.

The literature on this subject relates particularly to experiments with hypophysectomized *Rana*-larvae, in which a regression of the interrenal tissue has been observed (SMITH, 1920). After the administration of adrenocorticotrope hormone the adrenals soon recover (SMITH and SMITH, 1923; ATWELL, 1935).

Similar results have been obtained in adult Amphibians. According to ADAMS and BOYD (1933) the size of the nuclei of the interrenal cells of the adrenals, as well as their lipoid content, decreases in *Triturus viridescens*, and PORTO (1940) has described a hypofunction in the adrenal lipoid cells (the interrenal cells) after removal of the anterior lobe of the pituitary in *Bufo arenarum*. Following injections of extracts of the anterior lobes of the pituitaries of both toads and mammals these cells resume their normal function.

It is our intention to trace the regression of the adrenal cells after hypophysectomy in adult specimens of *Rana esculenta* during a period of 2—9 months following the operation. Before describing these changes, however, it may be useful, in view of the Konsuloff-test, which is used more and more for clinical purposes, to describe at length MIGHORST's method of hypophysectomy (MIGHORST et al., 1949). This method is somewhat simpler than that used by SERVANTIE et al. (1947) and other workers.

2. Material and methods.

On Oct. 11 and 13, 1948 more than 100 specimens of *Rana esculenta* were hypophysectomized in the following manner.

After the frog has been anaesthetized by means of saturated ether-vapour and has been laid on a cork-plate, its mouth is opened as widely as possible with the help of two curved pins, which are hooked behind the upper and lower jaw respectively. The pin holding the lower jaw is connected to a string, fastened on the cork-plate behind the animal. Then the mucous

membrane of the palate is cut open in the median line from A to A₁ (fig. 1) ¹⁾ and the margins of the cut are pulled aside. Next, the parasphenoidal bone is superficially incised with a pair of scissors at the points 1, 2 and 3, indicated in fig. 1, and, using a pair of strong tweezers, is

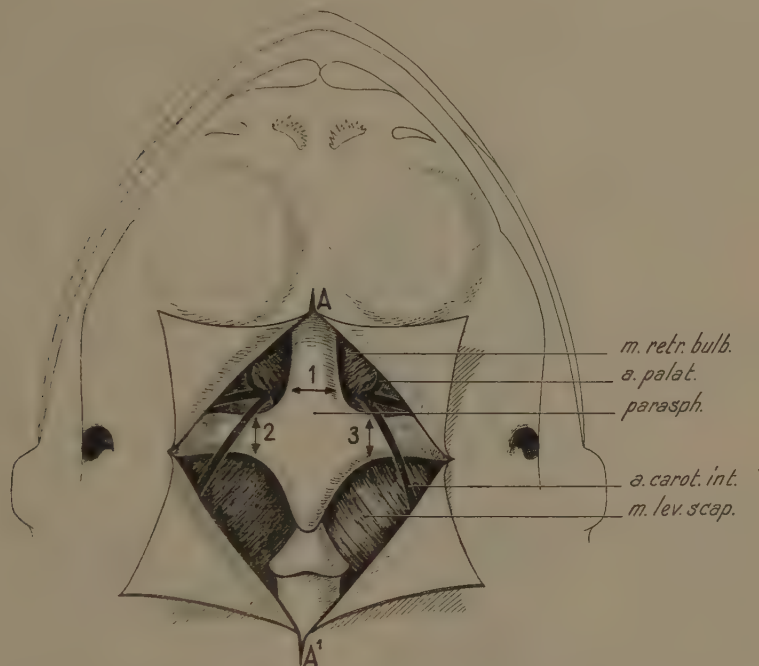


Fig. 1. *Rana esculenta*. The parasphenoidal bone, as seen before excision.

broken off here and at its joint with the first vertebra. The musculus retractor bulbi is cut, where it joins the parasphenoidal bone. Of course it is important not to damage the arteries.

Under a dissection microscope a horseshoe-shaped section (the opening of the horseshoe directed forward) is made in the cartilage, lying under the parasphenoidal bone. That part of the cartilage cut out is bent forward and kept in this position with the aid of a preparation needle. Now the brain is visible (fig. 2 in which, however, this cartilage is not shown). The entire pituitary is removed with the help of a small pipette (the diameter of which is of about the same size as the pituitary) and a narrow rubber tube by means of the adjustable sucking force of the operator's mouth.

After hypophysectomy the situation is as in fig. 3. The cartilaginous lid is then replaced in its natural position, and the wound in the mucous membrane in the palate is sewn up with surgical thread. Next the animals are placed separately into glass bowls with about 3 cm water, in which they rapidly recover.

¹⁾ To clarify the situation cuts perpendicular to line AA₁ are also drawn in fig. 1, but it is not necessary to make these cuts during the operation.

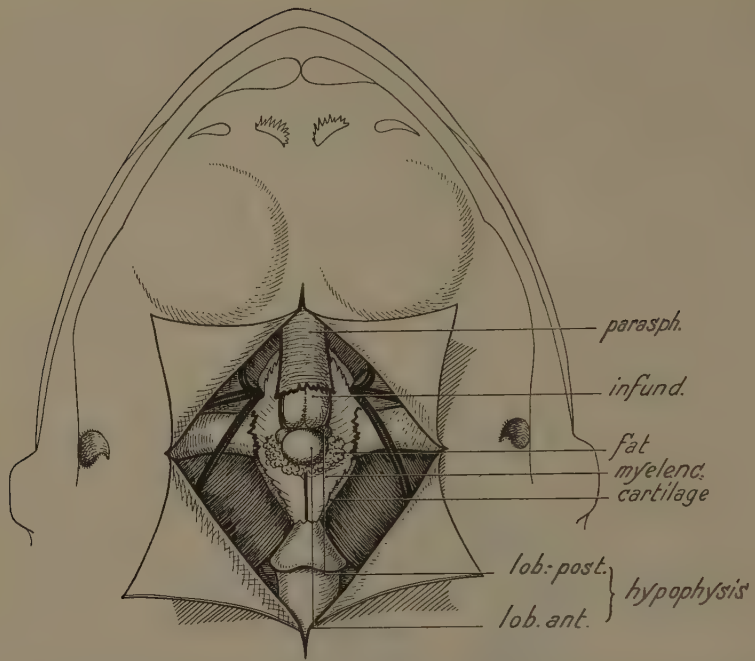


Fig. 2. *Rana esculenta*. The pituitary as seen after the removal of the parasphenoidal bone and the partial removal of the underlying cartilage.

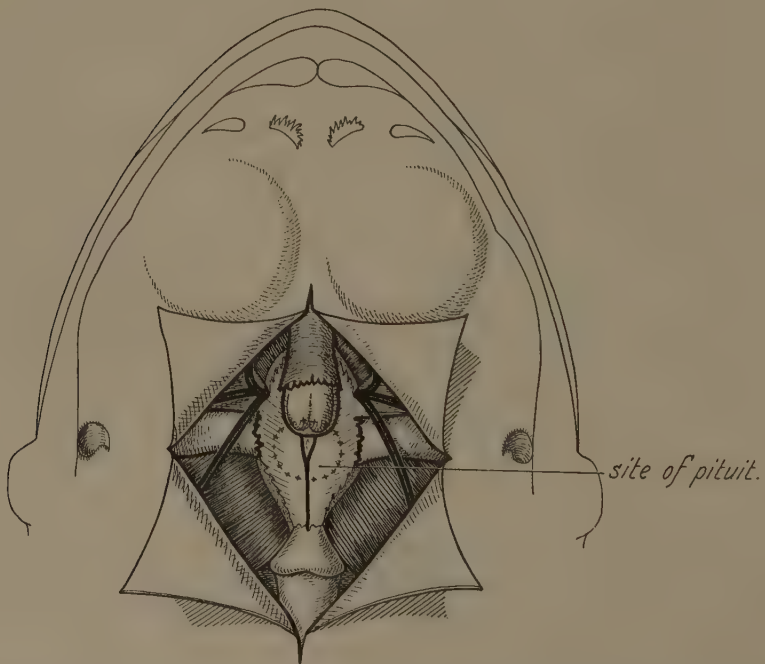


Fig. 3. *Rana esculenta*. The base of the brain as seen after hypophysectomy.

In *Rana esculenta* the mortality following the operation is about 10–20 per cent; during the breeding season, however, it is very high and often reaches 100 per cent. Thus, it is obvious that during the breeding period the pituitary has a much greater influence on the frog's body than at other times.

The hypophysectomized frogs were kept together with controls in a large vivarium at room temperature and fed copiously with mealworms, which were taken actively. Some experimental frogs were kept alive until July 1949 and so survived the operation for about 9 months. Experimentals and controls were killed at regular intervals and their adrenals fixed in Kolster's fluid, sectioned (3μ), and stained with Altmann's acid-fuchsin combined with brilliant-cresyl-blue.

3. Results.

The normal adrenal. The adrenal of the frog is composed of tissue strands which are separated by large blood lacunae. In these strands many interrenal and few suprarenal cells are to be found. Thus a true cortex and medulla are absent (fig. 4a).



Fig. 4. *Rana esculenta*. Cross sections of adrenal strands. The size of each fig. is approximately proportional to the size of the adrenals. Fig. 4a: Control-specimen, killed Dec. 7, 1949. Figs 4b and 4c: experimentals, hypophysectomized on Oct. 13, 1948, and killed respectively on Dec. 7, 1948 and May 23, 1949. Fixation: Kolster's fluid; Altmann-staining. ($\times 2000$.)

As in the higher vertebrates the interrenal cells are characterized by lipid droplets which fill these cells totally, while the suprarenal cells possess a chromophil plasm in which lipid droplets are lacking. In addition to these two cell types, so-called „summer-cells” are scattered in the frog's adrenal. These are filled with numerous fuchsinophil granules.

From October till July only insignificant fluctuations in the lipid content take place in the interrenal cells of frogs caught in the wild and of normal control frogs kept in confinement in the laboratory.

The adrenals following hypophysectomy. The first differences between adrenals of normal controls and of hypophysectomized frogs were visible about 2 months after the operation. On Dec. 7, 1948 some of the experimentals possessed adrenals in which several interrenal cells showed shrinkage of the nuclei and an almost total absence of lipid droplets, but the suprarenal and summer-cells were still normal (fig. 4b).

On Jan. 4, 1949 the nuclei of all 3 cell types in the experimental frogs were distinctly smaller than those of control specimens: lipid droplets were scarce and the whole gland had shrunk.

In March the adrenals showed no distinct deviations from those in January, but during May 1949 the regression of the adrenals had proceeded much further (fig. 4c). Then most of the cells possessed still smaller nuclei and still less protoplasm in which the mitochondria were strongly concentrated; lipid droplets were almost entirely lacking. Summer-cells and interrenal cells were difficult to distinguish from each other; the suprarenal cells were less affected.

The cytology of the adrenals in the experimental frogs in July was about the same as that of the adrenals in the experimental frogs in May. The cells were reduced in size, but nevertheless some still contained small numbers of lipid droplets. Consequently it is evident that the adrenal regression following hypophysectomy in October reached its maximum in May of the following year.

Summarizing, we can conclude that after hypophysectomy the cytology of the adrenals shows several typical features, suggesting a hypofunction of the cells. First, about 2 months after the operation the interrenal cells are affected; later the two other cell types (suprarenal and summer-cells) also show a distinct regression. Maximum adrenal regression does not occur until 7 months after hypophysectomy.

Acknowledgement. We wish to express our sincere thanks to Dr W. S. BULLOUGH (Sheffield) for correcting the English text of the present paper.

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Mathematics. — *On the number of terms of the square and the cube of polynomials.* By W. VERDENIUS. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 26, 1949.)

§ 1. *On the square of polynomials.*

A polynomial $\sum_{h=0}^n a_h x^h$ of the n^{th} degree is called complete if every coefficient a_h is different from zero.

We intend to prove in this section for every positive integer n the existence of a real complete polynomial $f(x)$ of the n^{th} degree such that $f^2(x)$ consists of less than $\frac{1}{7}(312^{13\log 8} - 18)$ terms, and moreover the existence of a complete polynomial $f(x)$ of the n^{th} degree with real integer coefficients such that $f^2(x)$ consists of less than $\frac{1}{5}(162 n^{9\log 6} - 12)$ terms.

We have

$$^{13}\log 8 = 0.810 \dots < ^9\log 6 = 0.819 \dots < 1.$$

As a special case we obtain the result, found by ERDÖS, that it is possible to construct for every positive integer n a real polynomial $f(x)$ with n coefficients $\neq 0$ such that $f^2(x)$ contains less than $c n^\gamma$ terms, where $\gamma < 1$ and c are absolute constants ¹⁾.

For the proof of our theorem we need three lemmas.

Lemma 1. *There exists a complete polynomial $P(x)$ of the 8th degree with integer coefficients, such that $P^2(x)$ does not contain the terms with exponent 2, 3, 4, 5 and 11, 12, 13, 14.*

Proof: If ²⁾

$$Q(x) = 1 + \frac{2}{\sqrt[7]{7}}x - \frac{2}{7}x^2 + \frac{4}{7\sqrt[7]{7}}x^3 - \frac{10}{49}x^4 + \\ + \frac{4}{7\sqrt[7]{7}}x^5 - \frac{2}{7}x^6 + \frac{2}{\sqrt[7]{7}}x^7 + x^8,$$

then $Q^2(x)$ does not contain the terms with the mentioned exponents. Consequently the polynomial $Q(x\sqrt[7]{7})$ satisfies all conditions of the lemma.

¹⁾ P. ERDÖS, On the number of terms of the square of a polynomial. *Nieuw Archief voor Wiskunde* 23, 63—65 (1948).

Cf. A. RÉNYI, On the minimal number of terms of the square of a polynomial. *Hungarica Acta Mathematica* 1, 30—34 (1947).

²⁾ This polynomial is already used by RÉNYI in his publication cited in the note ¹⁾.

Lemma 2. *There exists a real complete polynomial $P(x)$ of the 12th degree such that $P^2(x)$ does not contain the terms with exponent 3, 4, 5, 6, 7, 8 and 16, 17, 18, 19, 20, 21.*

Proof: It is sufficient to prove that the numbers $q_n(\lambda, \mu)$, defined by the expansion

$$\sqrt{1 + 2\lambda\mu x + \mu^2 x^2} = \sum_{n=0}^{\infty} q_n(\lambda, \mu) x^n,$$

satisfy for suitable chosen real λ and μ the relations

$$q_4(\lambda, \mu) = q_8(\lambda, \mu); q_5(\lambda, \mu) = q_7(\lambda, \mu); q_h(\lambda, \mu) \neq 0 \quad (h = 0, \dots, 6).$$

In fact, the polynomial

$$P(x) = \sum_{h=0}^5 q_h(\lambda, \mu) (x^h + x^{12-h}) + q_6(\lambda, \mu) x^h$$

satisfies the relation

$$\sqrt{1 + 2\lambda\mu x + \mu^2 x^2} = P(x) + x^9 Q(x),$$

where $Q(x)$ denotes a power series. Consequently

$$P^2(x) = 1 + 2\lambda\mu x + \mu^2 x^2 + x^9 R(x),$$

where $R(x)$ is a polynomial of degree 15 and in virtue of the symmetry of $P(x)$ its square contains neither a term with exponent ≥ 16 and ≤ 21 .

The numbers

$$\begin{aligned} q_4(\lambda, \mu) &= -2^{-3} \mu^4 (\lambda^2 - 1) (5\lambda^2 - 1); \\ q_5(\lambda, \mu) &= 2^{-3} \lambda \mu^5 (\lambda^2 - 1) (7\lambda^2 - 3); \\ q_7(\lambda, \mu) &= 2^{-4} \lambda \mu^7 (\lambda^2 - 1) (33\lambda^4 - 30\lambda^2 + 5); \\ q_8(\lambda, \mu) &= -2^{-7} \mu^8 (\lambda^2 - 1) (429\lambda^6 - 495\lambda^4 + 135\lambda^2 - 5) \end{aligned}$$

possess for each λ and μ the property

$$q_5^2 q_8 - q_4 q_7^2 = 2^{-13} \cdot 11 \mu^{18} \lambda^2 (\lambda^2 - 1)^5 \{64\lambda^6 + 5(\lambda^2 - 1)^3\}.$$

Consequently, if we choose $4\lambda^2 = 5^{\frac{1}{2}}(1 - \lambda^2)$, that is

$$\lambda = (1 + 4 \cdot 5^{-\frac{1}{2}})^{-\frac{1}{2}}$$

then we have for each μ the relation $q_5^2 q_8 = q_4 q_7^2$. In that case q_5 and q_7 have the same sign if $\mu \neq 0$, so that we can choose

$$\mu = \sqrt{\frac{q_5(\lambda, 1)}{q_7(\lambda, 1)}}.$$

Then we have

$$q_5(\lambda, \mu) = q_7(\lambda, \mu) \neq 0 \quad \text{and} \quad q_4(\lambda, \mu) = q_8(\lambda, \mu) \neq 0.$$

For that choice not only q_4 and q_5 , but also

$$q_0 = 1, q_1 = \lambda\mu, q_2 = -2^{-1}\mu^2(\lambda^2 - 1), q_3 = 2^{-1}\lambda\mu^3(\lambda^2 - 1)$$

and

$$q_6 = -2^{-4}\mu^6(\lambda^2 - 1)(21\lambda^4 - 14\lambda^2 + 1)$$

are different from zero.

A polynomial $g(x)$ is called periodic (of order 2) with the length h , when there exist h positive integers $n_1 < n_2 < \dots < n_h$ such that $g(x)$ contains, apart from the constant term, only terms with exponent $nt + rn_h$, where $t = 1, \dots, h$ and $r = 0, 1$.

Lemma 3. Let k be a positive integer³⁾ and $P(x)$ be a polynomial of degree $4k$, with the property, that in $P^2(x)$ do not occur the terms with exponent $k, k+1, \dots, 3k-1$ and $5k+1, \dots, 7k$. Suppose that the polynomial $g(x)$ is periodic with the length h and of degree $2n$. Then for each $\lambda \neq 0$ the polynomial $f(x)P^2(\lambda x^n)$ is periodic with the length $2h(k+1)+1$.

Proof: Let $g(x)$ contain, apart from the constant term, only terms with exponent $nt + rn_h$, where: $t = 1, \dots, h$; $r = 0, 1$ and $0 < n_1 < \dots < n_h = n$. Since in $P^2(x)$ the exponents $k, \dots, 3k-1$ and $5k+1, \dots, 7k$ do not occur, the polynomial $g(x)P^2(\lambda x^n)$ contains only terms with exponent un_h and $nt + (u+r)n_h$, where $t = 1, \dots, h$; $r = 0, 1$ and u belongs to the system S formed by the integers satisfying $0 \leq u \leq k-1$, $3k \leq u \leq 5k$ and $7k+1 \leq u \leq 8k$. This system S contains only 3 numbers v , such that $v-1$ does not occur in the system formed by the numbers $u+r$, viz. $v=0$, $v=3k$ and $v=7k+1$. Because $nt + (u+r)n_h = (u+r+1)n_h$ for $t=h$, the polynomial $f(x)P^2(\lambda x^n)$ obtains therefore only terms exponent $0, 3kn_h, (7k+1)n_h$ and

$$nt + wn_h (t=1, \dots, h; 0 \leq w \leq k, 3k \leq w \leq 5k+1, 7k+1 \leq w \leq 8k+1).$$

We divide the system of these numbers, apart from zero, in two systems, viz.

$$\begin{aligned} & 3kn_h, nt + w_1 n_h (t=1, \dots, h; 0 \leq w_1 \leq k, 3k \leq w_1 \leq 4k) \\ & (7k+1)n_h, nt + w_2 n_h (t=1, \dots, h; 4k+1 \leq w_2 \leq 5k+1, \\ & \qquad \qquad \qquad 7k+1 \leq w_2 \leq 8k+1). \end{aligned}$$

When we add to each number of the first system its largest number, that is $(4k+1)n_h$, we get the second system. The number of terms of each system is $2h(k+1)+1$. This completes the proof.

³⁾ We only need $k=2$ and 3 . When $k=1$ the result is not interesting because then $P(x)$ is not complete.

Proof of the theorem. Suppose $k = 2$ or 3 and let $P(x)$ be the polynomial of lemma 1 if $k = 2$ and that of lemma 2 when $k = 3$. Consider the polynomial

$$F_N(x) = (1+x)P(\lambda, x)P(\lambda_2 x^{4k+1}) \dots P(\lambda_N x^{(4k+1)^{N-1}}),$$

where N is a non-negative integer and $\lambda_1, \dots, \lambda_N$ are integers, all $\neq 0$, which will be chosen later.

We show by mathematical induction that $F_N(x)$ is for suitably chosen $\lambda_1, \dots, \lambda_N$ a complete polynomial of degree $(4k+1)^N$ and that $F_N^2(x)$ is periodic with the length

$$h = \frac{(2k+2)^{N+1} - 1}{2k+1}.$$

This is clear for $N = 0$. Suppose now that $N \geq 1$ and that this assertion is true when N is replaced by $N-1$. Because $F_{N-1}(x)$ is complete and of degree $(4k+1)^{N-1}$ it is possible to choose the integer λ_N in such a way, that $F_N(x)$ is complete. From lemma 1 and 2 we know that $P(x)$ is of degree $4k$. Consequently $F_N(x)$ is of degree $(4k+1)^N$. Since $F_{N-1}^2(x)$ is periodic with the length

$$\frac{(2k+2)^N - 1}{2k+1},$$

it follows from lemma 3 that $F_N^2(x)$ is periodic with the length h .

By means of this polynomial $F_N(x)$ we construct for every positive integer n a polynomial $f(x)$ of the n^{th} degree, whose square consists of less than

$$\frac{6(4k+1)(k+1)}{2k+1} n^{(4k+1)\log(2k+2)} - \frac{6k}{2k+1}$$

terms. If n is given, we define the four integers N, m, a and b , by

$$N = \lceil (4k+1)\log n \rceil, \quad m = (4k+1)^N,$$

and

$$n = am + b, \quad 1 \leq a \leq 4k, \quad 0 \leq b \leq m-1.$$

In the polynomial

$$f(x) = F_N(x) (1 + \mu_1 x^m + \dots + \mu_{a-1} x^{(a-1)m} + \mu_a x^{(a-1)m+b})$$

we can choose the integers $\mu_1, \dots, \mu_a$ all $\neq 0$, so that $f(x)$ is complete and of degree n , because $F_N(x)$ is complete and of degree m . By squaring we obtain

$$f^2(x) = F_N^2(x) \left(\sum_{h=0}^{2(a-1)} p_h x^{hm} + \sum_{h=a-1}^{2(a-1)} q_h x^{hm+b} + \mu_a^2 x^{2(a-1)m+2b} \right),$$

where p_h and q_h are integers. Because $F_N^2(x)$ is periodic it is easy to see that $f^2(x)$ consists of at most

$$((2a-1)h + h + 1) + (ah + h + 1) + (2h + 1)$$

terms. With

$$h = \frac{(2k+2)^{N+1} - 1}{2k+1} \equiv \frac{2(k+1)}{2k+1} n^{(4k+1)\log(2k+2)} - \frac{1}{2k+1}$$

and $1 \leq a \leq 4k$ we get the required upper bound for the number of terms of $f^2(x)$.

One final remark: If we should take in stead of $P(x)$ of lemma 1 or 2 a complete polynomial $P(x)$ of degree $4k$ with the properties mentioned in lemma 3, we could construct a complete polynomial of degree n in such a way that $f^2(x)$ consists of less than

$$c(k) n^{(4k+1)\log(2k+2)},$$

terms, where k denotes an integer ≥ 2 and $c(k)$ is a constant only depending on k . But since

$$13 \log 8 < (4k+1) \log(2k+2) \quad (k=2, 4, 5, \dots)$$

this result would not be very interesting:

§ 2. On the cube of polynomials.

We propose to prove the existence of two positive constants $\gamma < 1$ and c such that it is possible to construct for every positive integer n a complete polynomial $f(x)$ of the n th degree with the property that $f^3(x)$ consists of less than $c n^\gamma$ terms. Here "complete polynomial" has the same meaning as in § 1.

For the proof we need two lemmas.

Lemma 4. *There exists a complete polynomial $P(x)$ of degree $2k = 270$, such that the terms with exponents $k-2$, $k-1$, k , $3k-1$, $3k$, $3k+1$, $5k$, $5k+1$, $5k+2$ do not occur in $P^3(x)$.*

Proof: The coefficients of x^5 and x^6 in $Q^3(x)$, where

$$Q(x) = 1 + a_1 x + a_2 x^2 + a_1 x^3 + x^4,$$

are equal to $3a_1(a_1^2 + a_2^2 + 2a_2 + 2)$ and $a_2^3 + 6a_1^2 a_2 + 6a_1^2 + 6a_2$. Therefore $Q^3(x)$ does not contain the terms with exponents 5 and 6 if the coefficients $a_1 \neq 0$ and $a_2 \neq 0$ are so chosen that

$$a_1^2 + a_2^2 + 2a_2 + 2 = 0 \quad ; \quad a_2^3 + 6a_2 - 6(a_2 + 1)(a_2^2 + 2a_2 + 2) = 0.$$

That neither the term x^7 occurs follows from the symmetry of $Q(x)$. Similarly we find for the polynomial

$$R(x) = 1 + x - x^2 + bx^3 - 2bx^4 + bx^5 - x^6 + x^7 + x^8,$$

that the terms with exponents 2, 4, 11, 13, 20 and 22 do not occur in $R^3(x)$, if the coefficient $b \neq 0$ satisfies the equation

$$5b^3 + 5b^2 + 4b - 1 = 0.$$

The polynomial

$$P(x) = S(x) Q(\lambda x^6) R(\mu x^{30}),$$

where

$$S(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$$

and where λ and μ are so chosen, that $P(x)$ is complete, possesses the required properties. In fact, $Q^3(\lambda x^6)$ contains only terms with exponents 0, 6, 12, 24, 48, 54, 60, 66 and 72, so that the terms with exponent 43, 44, 45, 46 and 47 do not occur in $S^3(x)Q^3(\lambda x^6)$. As $R^3(\mu x^{30})$ does not contain a term with exponent $30h$, where $h = 2, 4, 11, 13, 20$ and 22 , the polynomial $P^3(x)$ does not contain a term with exponent 133, 134, 135, 136, 137, 403, 404, 405, 406, 407, 673, 674, 675, 676 and 677.

A polynomial $g(x)$ is called periodic (of order 3) with the length h , when there exist h positive integers $n_1 < n_2 < \dots < n_h$ such that $g(x)$ contains, apart from the constant term, only terms with exponent $n_t + r n_h$, where $t = 1, \dots, h$ and $r = 0, 1, 2$.

Lemma 5. Let $P(x)$ be a polynomial with the property mentioned in lemma 4. Suppose $g(x)$ is a periodic polynomial with the length h and degree $3n$. Then for each $\lambda \neq 0$ the polynomial $f(x) P^3(\lambda x^n)$ is periodic with the length $270h + 1$.

Proof: Suppose that $g(x)$ contains, apart from the constant term, only terms with exponent $n_t + r n_h$, where $t = 1, \dots, h$ and $r = 0, 1, 2$ and moreover $0 < n_1 < \dots < n_h = n$. Let $2k$ denote the degree of $P(x)$. By lemma 4 the polynomial $P^3(\lambda x^n)$ contains only terms with exponent $u n_h$, where u belongs to the system S , defined by $0 \leq u \leq 6k$, $u \neq k-2, k-1, k, 3k-1, 3k, 3k+1, 5k, 5k+1, 5k+2$. Consequently the polynomial $f(x) P^3(\lambda x^n)$ contains only terms with exponent $u n_h$ and $n_t + (u+r)n_h$, where $t = 1, \dots, h$; $r = 0, 1, 2$ and u belongs to S . This system S , contains only four numbers v such that $v-1$ does not occur in the system formed by the numbers $u+r$, viz. $v = 0, k+1, 3k+2$ and $5k+3$. Since $n_t + (u+r)n_h = (u+r+1)n_h$ for $t = h$, the polynomial $f(x) P^3(\lambda x^{n_h})$ contains therefore only the terms with exponent 0, $(k+1)n_h$, $(3k+2)n_h$, $(5k+3)n_h$ and $n_t + w n_h$ ($t = 1, \dots, h$; $0 \leq w \leq 6k+2$, $w \neq k, 3k+1, 5k+2$). The system formed by these exponents apart from zero can be divided in three systems, viz.

$$\begin{aligned} & (k+1)n_h, n_t + w_1 n_h \quad (t = 1, \dots, h; 0 \leq w_1 \leq 2k, w \neq k) \\ & (3k+2)n_h, n_t + w_2 n_h \quad (t = 1, \dots, h; 2k+1 \leq w_2 \leq 4k+1, w \neq 3k+1) \\ & (5k+3)n_h, n_t + w_3 n_h \quad (t = 1, \dots, h; 4k+2 \leq w_3 \leq 6k+2, w \neq 5k+2). \end{aligned}$$

It is clear that when we add to each number of the first system its largest number, that is $(2k+1)n_h$, we get the second and when it is added to each number of the second system we get the third. The number of elements in each system is clearly $2kh+1=270h+1$, and so the lemma is proved.

Proof of the theorem. Consider the polynomial

$$F_N(x) = (1+x)P(\lambda_1 x)P(\lambda_2 x^{271}) \dots P(\lambda_N x^{271^{N-1}}),$$

where N is a non-negative integer and $P(x)$ has the same meaning as in lemma 4.

As in § 1 it is easy to prove that $F_N(x)$ is for suitably chosen $\lambda_1, \dots, \lambda_N$ a complete polynomial of degree 271^N and $F_N^3(x)$ is periodic of the third order with the length

$$h = \frac{270^{N+1} - 1}{269}.$$

If n is given, we define the four integers N, m, a and b by

$$N = \lceil {}^{271}\log n \rceil, \quad m \equiv 271^N,$$

and

$$n = am + b, \quad 1 \leq a \leq 270, \quad 0 \leq b \leq m-1.$$

The polynomial $f(x) = F_N(x)G(x)$, where

$$G(x) = 1 + \mu_1 x^m + \dots + \mu_{a-1} x^{(a-1)m} + \mu_a x^{(a-1)m+b}$$

is for suitably chosen $\mu_1, \dots, \mu_a$ (all $\neq 0$) complete of degree n , because $F_N(x)$ is complete of degree m . Since $G(x)$ consists of at most 271 terms, its cube has less than 271^3 terms. Consequently the numbers of terms of $f^3(x)$ is less than $271^3(3h+1)$.

We choose $\gamma = {}^{271}\log 270 < 1$. In virtue of $270^N \leq n^{\gamma \log 270}$ we get that for suitable c the number of terms of $f^3(x)$ is less than $c n^\gamma$.

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